THE UNIVERSITY OF SUSSEX

Gravitational Properties of Quantum Bosonic Strings

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Submitted for the degree of D. Phil.

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DECLARATION

I hereby declare that this thesis has not been submitted, either in the same or different form, to this or any other University for a degree. However, the thesis incorporates in chapter 3 work done in collaboration with my supervisor Dr. E. J. Copeland and Prof. H. J. de Vega from the University of Paris and submitted for publication to the editors of "Physical Review D" and appears also in the list of preprints of Los Alamos National Laboratory under the preprint number:

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ABSTRACT

In this thesis we are interested in the study of the gravitational properties of quantum bosonic strings. We start by computing the quantum energy-momentum tensor $\hat{T}^{\mu\nu}(x)$ for strings in Minkowski space-time. We perform the calculation of its expectation value for different physical string states both for open and closed bosonic strings. The states we consider are described by normalizable wave-packets in the centre of mass coordinates. Amongst our results, we find in particular that $\hat{T}^{\mu\nu}(x)$ becomes a non-local operator at the quantum level, its position appears to be smeared out by quantum fluctuations. We find that the expectation value acquires a non-zero value for both massive and massless string states.

After computing $\langle\hat{T}^{\mu\nu}(x)\rangle$ we proceed to calculate the gravitational field due to a quantum massless bosonic string in the framework of a weak-field approximation to Einstein's equations. We obtain a multipole expansion for the weak-field metric $h^{\mu\nu}(x)$ and present its gravitational properties, including the gravitational radiation produced by such a string. Our results are then compared to those found for classical (cosmic) strings.

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CHAPTER 1

Introduction

One of the greatest challenges facing physics nowadays is the construction of a consistent theory of quantum gravity and thence a consistent theory for the unification of the fundamental forces in nature.

The last decades have witnessed a remarkable advance in the construction of a unified theory of the known fundamental interactions, namely: the electro-magnetic, the weak, the strong, and the gravitational forces. First, the electro-magnetic and weak forces were unified in the description given by the Weinberg-Salam theory [7, 8] and subsequently there have been important successes in incorporating the strong interactions, described by quantum chromodynamics, into a larger gauge theory [9]. However, despite all these successes gravity has remained the odd one out in this scheme of grand unification. Furthermore, here we are confronted with two problems: first, there is the problem of the actual unification of gravity with the other fundamental forces in a single grand unified theory and second the quantisation of gravity itself. We will see in section 1.2 that both problems are really encompassed in the second one.

1.1 Thesis

1.1.1 Gravitational properties

This thesis focusses mainly on the quantum aspects of fundamental bosonic strings in the framework of gravity. That is, we want to study the gravitational properties emerging from quantum strings.

The classical aspects of String Theory in this framework, *cosmic strings*, have been studied in numerous papers (see for example Vilenkin & Shellard [87] and references within). We would like to see how the quantum nature of strings affects some of the main results for cosmic strings. For example: is there an equivalent to the deficit angle obtained in the

classical theory?. The theory of fundamental strings, although mathematically equivalent to that of cosmic strings at the classical level, is very different in concept. Cosmic strings emerge when we break a gauge symmetry. Fundamental strings, on the other hand, are supposed to describe all the known (and unknown) particle fields in nature since String Theory can be seen as the most promising candidate for a quantum theory for the gravitational field. So how different is the gravitational field due to fundamental strings compared to the one due to a Cosmic string?. We will see that because of the quantum nature of the strings considered in this thesis, we will also be making contact with what is called a **semi-classical approach to quantum gravity**.

1.1.2 The expectation value of the string energy-momentum tensor

One very important aspect of our work is the computation of the string energy-momentum tensor in a truly stringy way, by this we mean that we will keep the extended nature of the string as opposed to other works where the string is integrated over a spatial volume and treated like a point particle [10]. Such a computation leads to the ordinary results for a point particle because the authors considered the string precisely to behave like a point particle; that is, they integrated over a spatial volume completely surrounding the string. In this way the calculation turns out to be very simple; however, if we do not integrate over a volume around the string, we keep all the string features and, as we will show in chapter 3, neither the calculation nor the results are simple anymore. The consequences and differences that arise in this approach will be presented in chapter 3. At the quantum level the string energy-momentum tensor in Minkowski space-time seems to emulate a vertex operator. This occurs not only because we are considering the string to be a quantum object but also because we are maintaining its string nature. As we will see, the gravitational field produced by fundamental strings gives very different results if we keep the string nature compared to that given by integrating over a volume embedding the string. A similar situation, although much more complicated, occurs for the calculation of the expectation value of the string energy-momentum tensor in curved space-times (for example in gravitational shock-wave space-times).

We have mentioned that our calculations also correspond to a semi-classical theory of quantum gravity. So it is convenient at this point to talk about what a quantum theory of gravity should be.

1.2 Theories of quantum gravity

What is a theory of quantum gravity? A quantum theory of gravity (assuming it exists) is a theory in which general relativity can be unified with quantum field theory. A quantum theory of gravity must necessarily be:

1. A finite theory (not in a renormalizable way but it has to be exactly finite).

2. A theory of everything.

The reason for this is as follows [11]: first let us recall what the meaning of a renormalizable quantum field theory (QFT) is. It is a theory that has a domain of validity characterised by energies E such that these energies remain below the scale of energies relevant to the model under consideration. That is

$$E < \Lambda,$$

where Λ is of the order of 1 GeV for QED, 100 GeV for the standard model of strong and electro-weak interactions, etc. One always applies QFT up to an infinite energy (equivalent to zero distances) for virtual processes and what we find in most cases is that the theory has ultra-violet divergences. These divergences reflect the fact that our model is unphysical for energies much larger than Λ. In a renormalizable QFT, these infinities can be absorbed in a finite number of parameters (like coupling constants and mass ratios). Since these parameters are not predicted by the model in question, they have to be fixed by their experimental values. This means that we need a more general theory valid at energies beyond Λ in order to compute these renormalized parameters. For example $\frac{M_W}{M_Z}$ can be computed in a *grand unified theory* whereas it must be fitted to its experimental value in the standard electro-weak model.

Now, let us analyse the consequences of Heisenberg's principle in quantum mechanics when it is combined with the notion of gravitational (Schwarzschild) radius in general relativity.

If we can make measurements at a very small distance Δx, then

$$\Delta p \sim \Delta E \sim \frac{1}{\Delta x}, \tag{1.1}$$

where we have set c=$\hbar$=1. For sufficiently large ΔE, particles with masses $m \sim 1/\Delta x$ will be produced. The gravitational radius of these particles will be of the order

$$r_G \sim Gm \sim \frac{(l_{Planck})^2}{\Delta x} \tag{1.2}$$

where $l_{Planck} \sim 10^{-33}cm$. General relativity allows us to make measurements at distances Δx provided

$$\Delta x > r_G \quad \rightarrow \quad \Delta x > \frac{(l_{Planck})^2}{\Delta x}, \tag{1.3}$$

which implies

$$\Delta x > l_{Planck} \quad or \quad m < M_{Planck}. \tag{1.4}$$

From the discussion above we can see that no measurements can be made at distances smaller than the Planck length and that there are no particles heavier than the Planck mass. This is a consequence of combining relativistic quantum mechanics with general relativity.

Since the Planck mass is the heaviest possible particle state, a theory valid in this domain has to be valid at any other lower energy scale. One may ignore higher energy phenomena in a low energy theory; but the opposite is not true. Therefore, we conclude that a theory valid at the energy scale of the Planck mass will be a *theory of everything.* It has to describe all known particle physics as well as the *desert* which exists beyond the standard model. A theoretical prediction for graviton-graviton scattering at energies of M_{Planck} must include all particles produced in a real experiment, which in practice means all existing particles in nature since all matter is coupled to gravity. It should be noted that the conclusion of a quantum theory of gravity being a theory of everything is independent of any model we want to study.

Once we have arrived at the conclusion above, it is clear that we must now conclude the following: a consistent theory of quantum gravity has to be a finite theory. In a quantum theory of gravity we have $\Lambda = M_{Planck}$ and there cannot be any theory of particles beyond it. Therefore if ultraviolet divergences appear in quantum gravity, there is no way to interpret them as coming from a higher energy scale as is done in QFT. No physical understanding can be given to such ultraviolet infinities. Therefore, the theory of quantum gravity has to be exactly finite and not renormalizable finite. Of course, in the discussion above we are assuming that both general relativity and quantum mechanics will hold at energies in the order of the Planck scale, something which it is far from obvious, and that we are dealing with a 3+1 space-time geometry.

The general theory of relativity is completely compatible with all other classical theories; however, the only observable classical fields are the electro-magnetic field and the

gravitational field. The many other interactions between the fundamental fields of nature can only be described properly with the aid of quantum mechanics. Hence the need to have a theory which includes all the interaction on an equal footing, a *grand unified theory.* Currently some of the most fruitful approaches to quantise the gravitational field are supergravity theories in which the graviton is regarded as one member of a multiplet of gauge particles including bosons and fermions [1]-[6]. And of course, the most serious candidate for not only quantising gravity but for actually being a theory of everything is *String Theory.*

1.2.1 The problems quantising the gravitation field

Let us first say something about why gravity seems not to be compatible with quantum mechanics. One of the main postulates of relativity is that a locally geodesic coordinate system can be introduced at every region of space-time (see fig.(1.1)) so the action of the gravitational field becomes locally ineffective and the space is approximately flat Minkowski space. Therefore, to say that in our neighbourhood, with its small curvature, the space is flat seems to be natural. This essential postulate is what in quantum mechanics is often taken for granted. This postulate also shows us the limitations of the coexistence between quantum theory and relativity theory: when we want to study physical processes in regions of space dominated by a strong curvature regime (close to singularities) and when one considers the behaviour of a far extended physical system, quantum mechanics and general relativity are no longer compatible with each other, the reason being that they start from different space structures. Quantum mechanics always presupposes a flat Minkowski space-time of infinite extent both in its fundamental commutator rules, which are formulated explicitly by the Lorentz group and in technical issues like expansion in plane waves, asymptotic behaviour at infinity or the formulation of conservation laws. Relativity theory tell us, however, that the space is Riemannian. Quantum mechanics is valid if we do not deviate too much from the Minkowski space-time. However in strong curvature regimes, the validity of quantum mechanics is far from obvious. Another contradiction arises from the idea of relativity theory that the propagators of space are the propagators of the interactions of the matter and can be measured out by material test bodies, including measurements over very small distances and with the metric in very small regions of space. If the dimensions of these regions are so small that atoms or elementary particles should be taken as objects, then their location is no longer so precisely

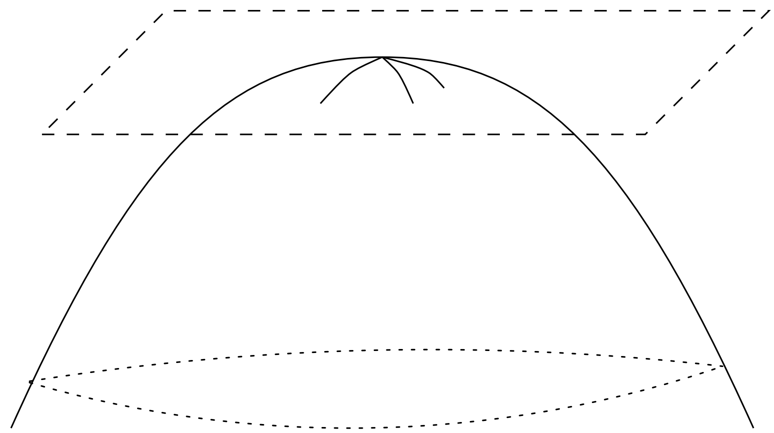

Figure 1.1: The space-time is locally flat, the postulates of quantum mechanics are valid as long as we do not deviate much from this picture.

defined and therefore we cannot speak of making any measurement.

Despite these observations, we know that there exist macroscopic bodies (e. g. stars) which consist of elementary particles and we know that the motion of these objects obey the laws of gravitation. Therefore, a consistent theory involving the merging of quantum mechanics and relativity should be possible, although it is expected that at least one of the two theories must be modified. In this direction, there are at least three possibilities towards unifying gravity with quantum mechanics. These will be presented in the following subsections. Before going into them, let us review briefly the 'standard' description of our Universe.

1.2.2 The standard Big-Bang model

The most cherished model that aims to explain the evolution of the Universe is the so called Big-Bang scenario. The Universe seems once to have been a great expanding *fire ball* of quarks, leptons and gluons existing at an enormous temperature.

The standard Big-Bang cosmology rests on three fundamental theoretical pillars: the cosmological principle, the theory of general relativity and a perfect fluid description of matter [36].

The cosmological principle

The cosmological principle states that on large distance scales the Universe is homogeneous [35]. From the observational point of view, this is a nontrivial statement. On small scales

the Universe looks extremely inhomogeneous: we can see with our naked eye that stars are not randomly, distributed. They are clustered in galaxies. Further observations tell us that galaxies are not randomly distributed either: galaxies also cluster into clusters of galaxies. Until recently every new survey has showed that there are new structures on the scale of the sample volume. In terms of the visible distribution of matter there seems to be no evidence for large scale homogeneity [14]. Recently there has been the discovery of considerable large-scale disturbances in the Hubble expansion in the neighbourhood of our Local Group of galaxies [15, 16, 17, 20, 21]. This has been claimed to be one of the most important results of observational cosmology in recent years [19]. It is not known in the present status of the theory, when most of the structure of a given scale formed and what most of this mass is. The large coherent flow of galaxies (including the Milky Way) suggests the presence of an extensive over density of matter that is now known with the name of *great attractor* [16]. Recent observations have shown that there seems to exist a more distant and greater attractor than the great attractor, the entire Pisces region is moving in the general direction of their local supercluster and in the direction of the great attractor; however, its peculiar velocity of around 400 Km/s is too fast even for the combined action of the gravitational pull from the local supercluster and the great attractor suggesting the presence of a larger mega-structure [18]. These observations are difficult to reconcile with the homogeneous picture of the Universe [14, 22]. In fact nowadays there is an ongoing debate regarding whether the Universe is homogeneous or not on large scale [13, 14]. However, the most compelling evidence for the homogeneity of the universe is given by the smoothness of the *cosmic microwave background radiation* (CMBR). Data from COBE indicates that the Universe is extremely homogeneous. Work done by [23, 24] strongly suggests that given the smallness of the anisotropies in the CMBR the cosmological principle is indeed valid.

The general theory of relativity

The second theoretical pillar, the theory of general relativity, is the theory which determines the dynamics of the Universe. According to general relativity, space-time is a smooth manifold. Together with the cosmological principle, this tells us that it is possible to choose a family of hypersurfaces with maximal symmetry. These are the homogeneous

constant time hypersurfaces. The metric of these surfaces is (in spherical coordinates) [98]

$$ds^2 = a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right]. \tag{1.5}$$

The constant k is $+1$, 0 or -1 for closed, flat or open surfaces respectively. The function $a(t)$ is the scale factor of the Universe. By a coordinate choice, it could be set equal to 1 at any given time. However, the time dependence of $a(t)$ indicates how the spatial sections evolve as a function of time. The full space-time metric is given by

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right]. \tag{1.6}$$

The most important consequence of general relativity for the history of the Universe is that it relates the expansion rate to the matter content in the following way: Hubble in 1929 found that the velocity of celestial bodies was somehow proportional to their relative distances, the relation between the velocity and the distance given by

$$v = \frac{\dot{a}(t)}{a(t)}\, d = H\, d$$

this is the so called 'Hubble's law'. In this relation H is 'Hubble's constant' (which as we can see, is not constant but varies as the Universe evolves in time). The value of this constant today is:

$$H_0 = 100\, h\, Km\ s^{-1}\ Mpc^{-1}$$

and h is believe to be in the range of $0.4 \leq h \leq 1$.

The relation between the expansion of the Universe and its matter content is given by the following equation of motion:

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 - \frac{8\pi G}{3}\rho = -\frac{k}{a^2(t)}.$$

Here the first term in the LHS is the kinetic energy term and the second one is the potential term. ρ is the matter density and k is a constant that can be positive, negative or zero depending on the matter density. If $k < 0$, the expansion dominates over the matter gravitational pull and the Universe will expand forever, this is called an *open Universe*; if $k > 0$, the matter term dominates over the expansion one and eventually the expansion ceases giving place then to a 'contraction' era, the Universe then ends in a 'big crunch', this Universe is called a *closed Universe*; finally, if $k = 0$ neither of the two terms dominates and we call this Universe a *flat Universe.* So far the bulk of observational evidence suggests that we live in an *open Universe*, although with the large percent of 'missing' matter (dark matter) still to find this view may change in the future.

Observational evidence

The three observational pillars of the standard Big-Bang cosmology are Hubble's redshift-distance relation, the existence of a black body spectrum of the cosmic microwave background, and the concordance between observed and theoretically determined light element abundances. The cosmic microwave background radiation observational data suggests that the Universe was extremely uniform at its infancy; therefore, in order to explain great attractor-like structures mentioned above, we may want to postulate that there were also primordial density variations which in time grew. Their gravitational pull then acted like a 'gravity amplifier' drawing more and more matter forming over dense regions (at the expense of under dense regions) of matter. However, this 'gravity amplifier' would not have been enough to turn the quantum fluctuations expected in the Big-Bang into today's galaxies and galaxy clusters let alone mega-structures like the great attractor (given the age of the Universe between 10 and 20 billion years old). However, if we consider inflation, the situation may improve.

The problems facing the standard Big-Bang cosmology

Standard Big-Bang cosmology is faced with several important and fundamental problems: the age, dark matter, homogeneity, flatness, and formation of structure problems. In addition the model does not explain the small value of the cosmological constant nor the very fundamental problem: What triggered the 'bang'?. Can we go backwards in time to times before the Big-Bang explosion? An answer to these questions is not possible within the current status of physical theories, although present data is consistent with the standard 'Big-Bang' model (if we consider that an epoch of inflation at the very early stages of the Universe indeed occurred).

Inflation

Because inflation is nowadays one of the most studied models for the description of the Universe we live in, it is worth to present here a few words about it.

Inflation is an epoch of near exponential expansion in the very early history of the Universe, in this way all the presently observable Universe comes from a tiny initial region of space. Models of inflation generally rely on the dynamics of a scalar field usually called the *inflaton*. At a certain 'early' time it is assumed that the inflaton is displaced from

the absolute minimum of its potential, the potential energy density dominates all other sources of energy density and leads to a period of exponential expansion in the Universe as time evolves until it no longer dominates as the inflaton reaches its absolute minimum [27, 28].

Inflation besides solving the horizon and entropy (flatness) problems, can predict an almost scale invariant initial spectrum of fluctuations consistent with the COBE data [29, 30]. These observations have been considered a great success for inflation. However, it is also possible to produce a similar spectrum of fluctuations from topological defects emerging from phase transitions in the early Universe (in particular from cosmic strings).

Another unresolved issue for inflation is that in most attempts to incorporate inflation into specific models of particle physics there exists at least two important problems: parameters such as coupling constants must be fine tuned to extremely small values in order to avoid overproduction of density fluctuations [25, 26].

1.2.3 The quantisation of space-time: The wave-function of the Universe

Let us go back now to the possibilities we were considering before regarding how it may be possible to unify gravity with quantum mechanics.

First, the physics in our classical Universe may well be very different from that of a quantum Universe. One possible way to unify gravity with the notions of quantum mechanics is to consider a Universe which is a mixture of states, each with an a priori probability of occurrence. Each state corresponds to a possible three geometry, including its topological properties, and can be described by a point in superspace. This approach involves the so called 'wave-function' of the Universe: Ψ. It is possible to define this wave-function by fixing the metric and other fields that may be present on a hypersurface Σ and then performing a path integral over bounded metrics and other fields. In this approach, the constants of nature can take on different values in different Universes resulting from: different choices of vacuum states, different compactifications or to more complicated phenomena such as worm-hole effects. It is to be also noticed that this approach is particularly sensitive to the initial conditions of the theory. However, we know that Ψ has to be unique; therefore, we need a law of boundary conditions [12]. The following are some of the boundary conditions that have been proposed in the literature [12]:

1. The Hartle-Hawking boundary condition [31]: they proposed that $\Psi(h, \phi)$ should be

given by a path integral over Euclidean 4-geometries $g^{\mu\nu}(\vec{x},t)$ and bounded by the 3-geometry $h_{ij}(\vec{x})$ in the following way:

$$\Psi(h,\phi) = \int [dg][d\phi]e^{-S_E}.$$

The wave-function describes basically a semi-classical tunnelling from 'nothingness' [39] to a Universe.

It should be noticed that in most attempts to compute the Hartle-Hawking wave-function, two important approximations are made [38]: 1) the path integral is evaluated by means of a saddle point approximation; 2) only the leading 'least' action extremum is taken into account.

With this proposal, Grishchuk and Rozhansky [41] computed the 'most likely' values of the scalar field ϕ that is predicted in this framework at the beginning of the Lorentzian stage in the evolution of the Universe. Their results show that the most probable values of ϕ are smaller than those which provide the minimally sufficient duration of inflation, concluding that such a wave-function does not describe in an adequate way the Universe we live in. However, the values they found are only smaller by a factor of 3 or 4 than the values needed, which still can make this wave-function approach consistent with inflation and the large scale structure of the observable Universe [40, 42].

2. Lorentz path integral (a proposal by Alex Vilenkin, see for example [32]): the wave-function of the Universe should be obtained by integrating over Lorentzian histories interpolating between a vanishing 3-geometry and a finite 3-geometry $h_{ij}(\vec{x})$

$$\begin{aligned} \Psi(h,\phi) &= \int [dg][d\phi]e^{iS_E} \\ &= K(h,\phi|0) \end{aligned}$$

where $K(h,\phi|0)$ is a causal propagator. In this scenario Vilenkin is able to obtain just the right initial conditions for inflation [32].

3. Linde's proposal [33]: He suggested that the Wick's rotation in the path integral should be done in the opposite direction $t \to +i\tau$

$$\Psi(h,\phi) = \int [dg][d\phi]e^{S_E}.$$

Here S_E is the Euclidean action given by [31]:

$$S_E = \int d^4x\sqrt{-g}\left(-R + \Lambda - \frac{1}{2}(\nabla\phi)^2 - \frac{R}{12}\phi^2\right).$$

R is the scalar curvature and Λ is a cosmological constant.

Another approach is given in [34] where the authors studied the wave-function of superstring theories in curved space-time. In doing that, some insight can be gained about the origin and evolution of the very early Universe. In particular they investigated the influences of the string vacuum fluctuations on processes occurring in the early Universe.

From a philosophical point of view, this approach of quantising the gravitational field via a wave-function may run into some problems. Since we are talking about a probability wave-function for the Universe, a randomly picked Universe may not be suitable for life.

1.2.4 Semi-classical gravity

A less radical approach to that of quantising gravity is to treat the gravitational field classically, but quantise all other fields. (This line of work is the one we will follow throughout this dissertation.) In a semi-classical theory, the coupling of gravity to the quantised field depends on the one hand on the fact that the field equations for gravity can be formulated covariantly, and therefore can be made to depend on the gravity field, and on the other hand on the fact that the gravitational field can be seen to have the quantum fields as its source. These fields occur, however, in the source of Einstein's field equations (the energy-momentum tensor) not as operators but as expectation values:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa\langle\hat{T}^{\mu\nu}\rangle. \tag{1.7}$$

And we need the further requirement that the expectation value of the energy-momentum tensor be divergence free in order for the field equations to be integrable. That is:

$$\langle\hat{T}^{\mu\nu}\rangle_{;\nu} = 0. \tag{1.8}$$

Here, as usual ';' means covariant differentiation. We can see that this expression is not a simple consequence of the equations governing the quantum fields, but rather a constraint on these quantities; for example, the states which are used to form the expectation values.

1.2.5 Quantum origins of the Universe

Another possibility for the unification of gravity with quantum mechanics is to consider how quantum mechanics actually allowed for the possibility for the existence of the Universe as we know it today.

The bouncing Universe

Can the Universe have begun with a nonsingular although very violent event? One possibility is considered in [43, 44], the Universe 'bounces'; there are terms in the effective action for gravity which are induced by quantum effects that reverse the collapse. If the matter density is enough, so that the Universe is closed, the question of origin need not arise; in some models these universes have always existed, eternally expanding and contracting.

Another possibility: a tunnelling effect

Another possibility [45] is that the Universe originated as a tunnelling effect from a classically stable, static space-time configuration. The Big Bang is analogous to a single radioactive decay, on a huge scale.

Studies in which people considered a quantum origin of the Universe began with the work of Tryon [46]. He suggested that the Universe might be a vacuum fluctuation; it began as nothing at all. If this is so, then the net quantum numbers of the Universe must be zero. In this respect, the total electric charge of the Universe is consistent with zero. The total baryon number is not consistent with zero. However, this is not so troubling since the grand unified theories of strong, weak, and electro-magnetic interaction imply proton instability. Tryon also adopts the view that the total energy must be strictly conserved in the creation process. This means that a Universe which originated as a vacuum fluctuation must have zero total energy.

Vacuum origin of the Universe

Brout, Englert, and Gunzig [47, 48] have further developed the idea of a vacuum origin of the Universe. They consider that the Universe was initiated by a *local* quantum fluctuation of the space-time metric. This results in particle creation. This creation of matter causes a further change in the metric, and a cooperative process is set up. During this fire ball stage of particle creation, which is characterised phenomenologically by negative pressure,

the Universe is an open de Sitter space-time which will develop a singularity, a future event horizon, within a finite proper time. Before this horizon is reached, however, the authors postulate that the cooperative process stops, and particle creation ends. Then begins the second stage of the evolution of the Universe: adiabatic free expansion with positive pressure, the usual post Big-Bang expansion. The authors examine in detail the particle creation mechanism and the joining of the fire ball and Big-Bang stages, but the origin of the quantum fluctuation is not examined.

1.2.6 String Theory

String Theory is the best candidate we have of a quantum theory of gravity. The way String Theory unifies the various interactions of nature is, roughly speaking, similar to the way a violin string gives a 'fundamental' description of musical sound. The musical notes are not the 'fundamental' entities; it is the violin string which is the 'fundamental' object. This object can give the description not only of the different tones that exist in music but also can give a description of full harmonies, which are constructed of different musical tones. In the case of String Theory, each 'note' may be interpreted as the different particles and forces that exist in nature, the string here being, as in the case of the violin, the fundamental object.

Superstring theory has no anomalies. The huge symmetry of the theory makes it possible to cancel all the potential anomalies the theory could have. Furthermore, it is hoped that the theory will remain finite to all orders in perturbation theory. An important point to stress here is that all these nice properties of String Theory are very sensitive to the string background we choose; therefore, there is very limited freedom in the theory. A consequence of this is that there are not so many free parameters involved in String Theory as we find in the GUT's. However, String Theory is not free of problems, the main problems that String Theory faces are the following:

1. First of all, the high energy region of the theory seems to be untestable since the energies involved are those found at the Planck scale (10^{19} GeV.). Obviously, a theory that cannot be tested is not an acceptable physical theory. Of course, there is hope that somehow low energy effects may arise from String Theory. There have been many studies on superstring phenomenology (see for example [49]-[51]) but its predictions are not well understood yet, this is in part due to the many ways we can

break the theory to low energies. Recently further advances have been made in order to test the low energy sector of superstring theory by investigating the possibility of spontaneous breaking of CP symmetry [58]. In the future we may hope to have indeed a way to test the theory.

2. The bosonic string theory has problems because of the appearance of a tachyon in the theory and because it does not include fermions. We can incorporate fermions into the theory by introducing supersymmetry. However, we still do not have direct evidence to confirm the existence of supersymmetry.

3. The theory seems to contain the general theory of relativity; however, it does not explain why the cosmological constant is zero.

4. The theory has thousands of ways to break-down to low energy. We do not know, therefore, which one is the correct vacuum for the theory. However, with the advent of M-theory the realisation of a *united* superstring picture seems to be closer. M-theory tells us that what was believe to be different superstring theories may actually be related to each other [81]. This has been one of the most important results in String Theory lately.

5. We do not know how to break the 26 dimensional bosonic string theory (10 dimensional in the case of superstrings) down to 4 dimensions in a dynamical way (in some analogy to the way of how we break the symmetry spontaneously in field theory), although there exists many ways to actually reduce the theory to 4 dimensions (see for example [49], [52]-[57]). This difficulty is the most fundamental one and therefore of paramount importance in String Theory.

CHAPTER 2

A brief introduction to String Theory

In this chapter, I will try to give a brief introduction to some of the most important issues regarding String Theory. The material covered here, however, has been selected on the basis of the work I will be developing in this thesis, mainly to try to present this work in a self-contained fashion.

2.1 Background

Today's physics is based on two fundamental theories. On the one hand we have the theory of general relativity, which has proved to be very successful in explaining not only the behaviour of the cosmos on large scales but also in leading to a sensible description of the behaviour of the Universe itself; on the other hand we have quantum field theory, which explains the physics of the microcosmos. Quantum field theory has been particularly successful in describing the weak and electro-magnetic interactions and some remarkable advances have been made in the quantum field description of strong interactions (Quantum Chromodynamics). Between the poles of these two theories all the present knowledge of physics is covered; that is, we have a description of nature for over 40 orders of magnitude.

Probably the most important goal in the minds of physicists these days yet to be achieved is to present a unified picture of all forces known in nature: the short-range forces, *strong* and *weak*; and the long-range forces, *electro-magnetic* and *gravitational*. The strong, weak and electro-magnetic forces can be described fairly well in the realm of quantum field theory whilst the gravitational one can be described by means of general relativity. However, up to this day the theory of general relativity has proved to be incompatible with quantum field theory; therefore, the ultimate goal of incorporating gravity together with the other interactions in a single theory has not been realised. String theories emerged in the late sixties, when G. Veneziano postulated his *Beta*-function amplitude

for strong interactions. In the 1960's the main problem facing physicists was the enormous proliferation of strongly interacting particles. One characteristic of these particles is that they seemed to have a spin proportional to the $mass^2$ of the particle. A theory of quantum fields consistent with higher spins is still lacking to this day. Another intriguing characteristic was that the scattering amplitudes seemed to have a duality in the s and t-channels. With this in mind, Veneziano constructed an amplitude which had precisely this dual behaviour [62]. Later on, it was discovered that behind this amplitude was really a relativistic string [74]. Dual models constructed from this idea give one way to incorporate particles of high spin without having ultraviolet divergences [60]-[74]. However, crucial developments in the seventies showed that these theories were not the right way of describing strong interactions. In particular the failure of dual models to incorporate the parton-like behaviour of strong interactions was the main reason for abandoning such models. Another problem with the dual model is that they predicted massless particles which experimentally are not observed in the strong interactions. One of these massless particles had a spin equal to 2. The coupling of this particle was similar to the ones of general relativity. This particle is now interpreted as a *graviton*. So now the view regarding dual models has changed: the possibility of treating these models as a theory for quantum gravity has given rise to a re-emergence of String Theory, not in the context of strong interactions but in the wider context of a possible theory incorporating gravity along with the other interactions of nature. At present the most promising hope for a truly unified and finite description of quantum field theory and general relativity is *Superstring Theory*. This may be due mainly to the huge set of gauge symmetries the theory possesses. We have to recall that all the advances in unifying the strong, weak and electro-magnetic interactions have been made thanks to the discovery of gauge symmetries.

2.2 Free point-particles

Before committing ourselves to work with extended objects such as strings, let us recall some of the aspects of the physics of point-particles. The notation we will follow throughout the chapters of this thesis will be as follows: greek indexes (μ, $\nu \ldots$) will run from 0 to $D-1$ where D are the number of space-time dimensions. Latin indexes (i, $j \ldots$) will run from 1 to $D-1$, that is, they will represent the *spatial* dimensions of our system. The metric will have signature (+,-,-,-,...).

Let us start now by considering a free spinless particle with mass m. It is clear that such a particle will follow a trajectory with only one parameter. Let us denote this trajectory by $X^\mu(\tau)$. This trajectory is usually known by the name of *world-line*. The classical action describing this particle must be independent of how the trajectory is parametrised and it is defined to be proportional to the arc-length travelled by the particle

$$S = -m \int_{s_i}^{s_f} ds = -m \int_{\tau_i}^{\tau_f} d\tau \sqrt{\dot{X}^2(\tau)}. \tag{2.1}$$

Where $\dot{X} = dX/d\tau$ and τ is an arbitrary parameter, that label points along the world-line. As we have said above, this action must be reparametrization invariant so let us show that that is precisely the case here. Let us make a change of coordinates from τ to $\tilde{\tau}$:

$$\tau \longrightarrow \tilde{\tau}(\tau)$$

With this transformation we obtain

$$d\tau = \frac{d\tau}{d\tilde{\tau}} d\tilde{\tau}$$

$$\frac{dX}{d\tau} = \frac{dX}{d\tilde{\tau}} \frac{d\tilde{\tau}}{d\tau}$$

therefore,

$$m \int d\tau \left[\left(\frac{dX}{d\tau} \right)^2 \right]^{1/2} = m \int d\tilde{\tau} \left[\left(\frac{dX}{d\tilde{\tau}} \right)^2 \right]^{1/2}.$$

Thus the action is invariant under arbitrary reparametrizations of the variable τ.

Introducing canonical conjugates:

$$P^\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -\frac{m \dot{X}^\mu}{\sqrt{\dot{X}^2}}$$

we see that not all the canonical momenta are independent. (Here the Lagrangian is given by $L = -m\sqrt{\dot{X}^\mu(\tau)\dot{X}_\mu(\tau)}$). There is the constraint

$$P^2 - m^2 \equiv 0. \tag{2.2}$$

This is a mass shell condition.

Now let us introduce the following Lagrangian with the help of a Lagrange multiplier e:

$$L = P_\mu \dot{X}^\mu - \frac{1}{2} e (P^2 - m^2). \tag{2.3}$$

By varying this Lagrangian with respect to e, we recover our constraint on the momenta. Let us now perform a path integral integrating functionally over the variable P

$$\int DP \exp\left\{i\int d\tau[P\dot{X} - \frac{1}{2}e(P^2 - m^2)]\right\}$$
$$\sim \exp\left\{i\int d\tau\frac{1}{2}(e^{-1}\dot{X}^2 - em^2)\right\}. \tag{2.4}$$

If in this expression we regard $X^\mu(\tau)$ as a set of scalar fields in one dimension, we can couple them to a metric $g \equiv g_{\tau\tau}$ and we can then write an action in a way similar to that in which we can couple scalar fields to gravity in four dimensions

$$S = -\frac{1}{2}\int d\tau\sqrt{-g}(g^{-1}\dot{X}^2 - m^2). \tag{2.5}$$

Here we can see that the mass term behaves like a cosmological constant. We can then vary this action with respect to the metric to obtain the equation of motion for g. Solving this equation of motion and substituting the solution in eq.(2.5), we obtain our previous action eq.(2.1). We can conclude that both actions are equivalent at least at the classical level. One important difference between these two actions is the fact that in eq.(2.5) we can actually take the limit of $m = 0$. Furthermore, we have eliminated the square root of the original action so now we have a linearised action which is easier to handle than the original one.

2.3 Free bosonic strings

2.3.1 The Nambu-Goto string action

The simplest extended object we can think of, built up from a single material point, is a one dimensional object: *a string*. Just as we associate to a point of matter some mass m (which may be zero), we can associate to each of the points of a string a *tension* $T = 1/2\pi\alpha'$ where α' is called the *Regge slope* parameter. In this sense, we may regard the string as a distinguishable collection of points in space-time, distinguishable precisely due to the tension associated to each of its points [78].

The string action is now proportional to the area the string generates as it moves. This area is usually called the *world-sheet* of the string. This action is given by [75]

$$S = -\frac{1}{2\pi\alpha'}\int d\sigma d\tau\sqrt{(\dot{X}^\mu X'_\mu)^2 - \dot{X}^2X'^2} \tag{2.6}$$

and is called the Nambu-Goto action of the string. Here σ and τ are arbitrary parameters which label points of the world-sheet. The prime denotes differentiation with respect to σ and the dot differentiations with respect to τ.

In the case of open strings it is necessary but not sufficient to ensure that the action is invariant under a general transformation

$$X^\mu \to X^\mu + \delta X^\mu.$$

The variation of the action under this transformation involves a term proportional to the wave equation and also the following surface term:

$$-\frac{1}{2\pi\alpha'}\int d\tau \left[X'_\mu\,\delta X^\mu\,\Big|_{\sigma=\pi} - X'_\mu\,\delta X^\mu\Big|_{\sigma=0}\right] = 0.$$

It is the vanishing of this surface term which gives the open string boundary condition. For closed strings the wave equation and periodicity of X is necessary and sufficient to ensure that the action is stationary, as we can see from the calculation below.

Let us consider an arbitrary change in the configuration of the string

$$X^\mu(\sigma,\tau) \to X^\mu(\sigma,\tau) + \delta X^\mu(\sigma,\tau).$$

We can now calculate the change in the action from the change in the Lagrangian [76, 78],

$$\begin{aligned}
\delta\mathcal{L}(\dot{X}^\mu, X'^\mu) &= \frac{\partial\mathcal{L}}{\partial\dot{X}^\mu}\delta\dot{X}^\mu + \frac{\partial\mathcal{L}}{\partial X'^\mu}\delta X'^\mu = \frac{\partial\mathcal{L}}{\partial\dot{X}^\mu}\frac{\partial}{\partial\tau}\delta X^\mu + \frac{\partial\mathcal{L}}{\partial X'^\mu}\frac{\partial}{\partial\sigma}\delta X^\mu \\
&= \frac{\partial}{\partial\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{X}^\mu}\delta X^\mu\right) + \frac{\partial}{\partial\sigma}\left(\frac{\partial\mathcal{L}}{\partial X'^\mu}\delta X^\mu\right) - \delta X^\mu\left[\frac{\partial}{\partial\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{X}^\mu}\right) + \frac{\partial}{\partial\sigma}\left(\frac{\partial\mathcal{L}}{\partial X'^\mu}\right)\right]
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
\delta S &= -\frac{1}{2\pi\alpha'}\int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma\delta X^\mu(\sigma,\tau)\left[\frac{\partial}{\partial\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{X}^\mu}\right) + \frac{\partial}{\partial\sigma}\left(\frac{\partial\mathcal{L}}{\partial X'^\mu}\right)\right] \\
&\quad + \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma\,\frac{\partial\mathcal{L}}{\partial\dot{X}^\mu}\delta X^\mu\Big|_{\tau_i}^{\tau_f} + \frac{1}{2\pi\alpha'}\int_{\tau_i}^{\tau_f} d\tau\,\frac{\partial\mathcal{L}}{\partial X'^\mu}\delta X^\mu\Big|_0^\pi.
\end{aligned} \tag{2.8}$$

From expression (2.8)we can obtain the boundary conditions mentioned above as well as the equations of motion for the string. (The open string is conventially described by the parameter σ running from 0 to π; for closed strings the same convention is used.) The equations of motion are given by

$$\frac{\partial}{\partial\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{X}^\mu}\right) + \frac{\partial}{\partial\sigma}\left(\frac{\partial\mathcal{L}}{\partial X'^\mu}\right) = 0, \tag{2.9}$$

and the boundary conditions are given by

$$\int_{\tau_i}^{\tau_f} d\tau\ \frac{\partial \mathcal{L}}{\partial X'^{\mu}} \delta X^{\mu}\bigg|_{\pi} - \int_{\tau_i}^{\tau_f} d\tau\ \frac{\partial \mathcal{L}}{\partial X'^{\mu}} \delta X^{\mu}\bigg|_{0} = 0. \tag{2.10}$$

That is, for open strings

$$\frac{\partial \mathcal{L}}{\partial X'^{\mu}} \delta X^{\mu}\bigg|_{\pi} = \frac{\partial \mathcal{L}}{\partial X'^{\mu}} \delta X^{\mu}\bigg|_{0} = 0 \tag{2.11}$$

and

$$\frac{\partial \mathcal{L}}{\partial X'^{\mu}} \delta X^{\mu}\bigg|_{\pi} = \frac{\partial \mathcal{L}}{\partial X'^{\mu}} \delta X^{\mu}\bigg|_{0} \tag{2.12}$$

for closed strings. For open strings this is clear from eq.(2.10). For closed strings this is due to the periodicity $\delta X^{\mu}(\pi, \tau) = \delta X^{\mu}(0, \tau)$.

Now we can define the conjugate momenta to σ and τ as [75, 78]

$$P^{\mu}_{\tau}(\sigma, \tau) \equiv -\frac{\partial \mathcal{L}}{\partial \dot{X}^{\mu}} = \frac{1}{2\pi\alpha'} \frac{(\dot{X}^{\rho} X'_{\rho}) X'^{\mu} - X'^2 \dot{X}^{\mu}}{\sqrt{(\dot{X}^{\rho} X'_{\rho})^2 - X'^2 \dot{X}^2}} \tag{2.13}$$

and

$$P^{\mu}_{\sigma}(\sigma, \tau) \equiv -\frac{\partial \mathcal{L}}{\partial X'^{\mu}} = \frac{1}{2\pi\alpha'} \frac{(\dot{X}^{\rho} X'_{\rho}) \dot{X}^{\mu} - \dot{X}^2 X'^{\mu}}{\sqrt{(\dot{X}^{\rho} X'_{\rho})^2 - X'^2 \dot{X}^2}}. \tag{2.14}$$

Thus the equations of motions (2.9) become

$$\frac{\partial P^{\mu}_{\tau}}{\partial \tau} + \frac{\partial P^{\mu}_{\sigma}}{\partial \sigma} = 0,$$

that is

$$\frac{\partial P^{\mu}_{\alpha}}{\partial \sigma^{\alpha}} = 0 \tag{2.15}$$

where $(\sigma^{\alpha} = (\tau, \sigma))$. Notice that the equations of motions obtained here are very complicated. This is due to the fact that we have not chosen any gauge yet.

2.3.2 The Polyakov string action

As in the point-particle case, we can express this action in a more convenient way, which is totally equivalent to the Nambu-Goto action of the previous section. The Polyakov form of the action is given by [64, 76]

$$S = -\frac{1}{2\pi\alpha'} \int d\sigma d\tau \sqrt{h} h^{\alpha\beta}(\sigma) g_{\mu\nu}(X) \partial_{\alpha} X^{\mu} \partial_{\beta} X^{\nu}, \tag{2.16}$$

where $h^{\alpha\beta}$ $(\alpha,\, \beta = 0, 1)$ is the inverse of $h_{\alpha\beta}$ and h is the absolute value of the determinant of $h_{\alpha\beta}$. The metric $h_{\alpha\beta}$ is a Minkowskian metric in 1+1 dimensions. The string coordinates

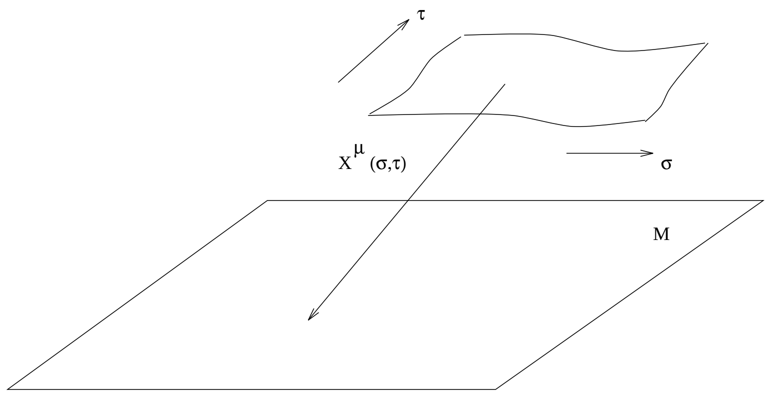

Figure 2.1: String world-sheet.

$X^\mu(\sigma,\tau)$ give a map of the world-sheet manifold into physical space-time (fig.2.1). The reparametrization invariance of this action allows us to make a covariant gauge choice, namely:

$$\sqrt{h}h^{\alpha\beta} = \eta^{\alpha\beta}, \tag{2.17}$$

where $\eta^{\alpha\beta}$ is of course a two-dimensional Minkowski metric. This gauge gives the following action:

$$S = -\frac{1}{2\pi\alpha'}\int d\sigma d\tau g_{\mu\nu}(X)\partial_\alpha X^\mu \partial^\alpha X^\nu, \tag{2.18}$$

and the constraints:

$$(\partial_\tau X^\mu \pm \partial_\sigma X^\mu)^2 = 0. \tag{2.19}$$

This is equivalent to saying that the energy-momentum tensor on the world-sheet must vanish in this gauge. The gauge choice we have made is known as the *conformal gauge* [59]. In the conformal gauge we find the wave equation

$$\Box X^\mu = \left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right) X^\mu = 0. \tag{2.20}$$

We can write the solution as a sum of two arbitrary left and right moving wave functions

$$X^\mu(\sigma,\tau) = X^\mu_R(\tau-\sigma) + X^\mu_L(\tau+\sigma). \tag{2.21}$$

X^μ_R here describes the right-moving modes of the string whilst X^μ_L describes the left-moving ones. Thus we can see that the string coordinates are given by

$$X^\mu(\sigma,\tau) = q^\mu + 2\alpha' p^\mu \tau + i\sqrt{\alpha'}\sum_{n\neq 0}^{\infty}\frac{e^{-in\tau}}{n}\alpha^\mu_n \cos n\sigma \tag{2.22}$$

for open strings and

$$X^\mu(\sigma,\tau) = q^\mu + 2\alpha' p^\mu \tau + i\sqrt{\alpha'} \sum_{n\neq 0}^{\infty} \frac{e^{-in\tau}}{n} \left(\alpha_n^\mu e^{in\sigma} + \tilde{\alpha}_n^\mu e^{-in\sigma} \right) \tag{2.23}$$

for closed strings.

Here, $q^\mu + 2\alpha' p^\mu \tau$ describes the centre of mass coordinates and the α_n^μ and $\tilde{\alpha}_n^\mu$ describe the right and left oscillation modes of the string respectively. In order to simplify the notation, we will from here onwards substitute α' for its value $1/2$[1].

Covariant quantisation gives the following commutation relations:

$$\begin{aligned} \left[X^\mu(\sigma,\tau), P_\tau^\nu(\sigma',\tau)\right] &= -i\eta^{\mu\nu}\delta(\sigma-\sigma'), \\ \left[X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\right] &= 0, \\ \left[P_\tau^\mu(\sigma,\tau), P_\tau^\nu(\sigma',\tau)\right] &= 0, \end{aligned} \tag{2.24}$$

which in turn yield the commutation relations for the operators α^μ and $\tilde{\alpha}^\mu$.

$$\begin{aligned} [\alpha_m^\mu, \alpha_{-n}^\nu] &= \quad -m\delta_{m,n}\eta^{\mu\nu}, \qquad n,\ m > 0 \\ [\alpha_m^\mu, \alpha_n^\nu] &= \quad [\alpha_{-m}^\mu, \alpha_{-n}^\nu] = 0, \quad n,\ m > 0 \\ [\alpha_m^\mu, \tilde{\alpha}_n^\nu] &= \quad 0 \\ [\tilde{\alpha}_m^\mu, \tilde{\alpha}_{-n}^\nu] &= \quad -m\delta_{m,n}\eta^{\mu\nu}, \qquad n\ m > 0, \\ [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] &= \quad [\tilde{\alpha}_{-m}^\mu, \tilde{\alpha}_{-n}^\nu] = 0, \quad n,\ m > 0. \end{aligned} \tag{2.25}$$

At the quantum level, the α_n are related to the familiar normalised harmonic oscillator operators by

$$\begin{aligned} \alpha_n^\mu &= \sqrt{n} a_n^\mu, \quad n > 0 \\ \alpha_{-n}^\mu &= \sqrt{n} a_n^{\mu\dagger}, \quad n > 0, \end{aligned} \tag{2.26}$$

with similar expressions for the $\tilde{\alpha}^\mu$. Here the α_n^μ and $\tilde{\alpha}_n^\mu$ are annihilation operators whilst the α_{-n}^μ and $\tilde{\alpha}_{-n}^\mu$ are creation operators.

[1] There are two conventions for the value of α'. For open strings $\alpha' = 1$ whilst for closed strings $\alpha' = 1/2$. Many of the results in String Theory are independent of whether we are working with closed strings or open strings so we will indicate which convention is followed only when the result is related to the open or closed string.

2.4 The Virasoro algebra

The Fock space is built up by applying the raising operators α^{μ}_{-n} to the ground state $|0\rangle$. This Fock space is not positive definite since the time components of these operators have a minus sign in their commutation relations, $[\alpha^0_n, \alpha^0_{-n}] = -n$, and therefore a state of the form $\alpha^0_{-n}|0\rangle$ has negative norm since $\langle 0|\alpha^0_n \alpha^0_{-n}|0\rangle = -n$. From this discussion we may see that the physical space of allowed string states consists of a subspace of the Fock space. This subspace is specified by certain subsidiary conditions. For instance, in the classical theory the vanishing of the world-sheet energy-momentum tensor represents the subsidiary conditions. The Fourier modes of the world-sheet energy-momentum components give the Virasoro generators:

$$L_m = -\frac{1}{4\pi}\int_{-\pi}^{\pi} e^{im\sigma}\left(\dot{X} + X'\right)^2 d\sigma = -\frac{1}{2}\sum_{-\infty}^{\infty} \alpha_n \cdot \alpha_{m-n}, \tag{2.27}$$

as well as a similar expression $\tilde{L}_m$ in the case of closed strings.

Because of a normal ordering ambiguity arising from the expression for $m = 0$, we include an undetermined constant a and then we say that a physical state $|\phi\rangle$ must satisfy

$$(L_0 - a)|\phi\rangle = 0. \tag{2.28}$$

For closed string we have in addition the following condition:

$$(L_m - \tilde{L}_m)|\phi\rangle = 0. \tag{2.29}$$

Equations (2.28) and (2.29) determine the mass of a string state in terms of its internal state of oscillation. We can see this from eq.(2.27) as follows:

$$-\frac{1}{2}\,(\alpha^{\mu}_0)^2 - \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n - a = 0, \tag{2.30}$$

Here α^{μ}_0 and $\tilde{\alpha}^{\mu}_0$ are identified with the string momentum p^{μ} in the following way:

$$(\alpha^{\mu}_0)^2 = p^{\mu}p_{\mu}$$

for open string, and

$$(\alpha^{\mu}_0)^2 = (\tilde{\alpha}^{\mu}_0)^2 = \frac{1}{4}p^{\mu}p_{\mu}$$

for closed string. So substituting back in expression (2.30) we obtain:

$$\frac{1}{2}p^{\mu}p_{\mu} + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n + a = 0, \tag{2.31}$$

for open strings, and

$$\frac{1}{8}p^{\mu}p_{\mu} + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n + a = 0, \tag{2.32}$$

for closed strings. With these expressions we obtain the mass shell condition for open and closed strings:

$$M^2 = -2a - 2\sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n, \tag{2.33}$$

in the open string case; whilst for the closed string case we have:

$$M^2 = -8a - 8\sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n = -8a - 8\sum_{n=1}^{\infty} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n. \tag{2.34}$$

Expressions (2.33) and (2.34) show that the bosonic string ground state has a negative mass squared; that is, the ground state of a bosonic string is a *tachyon*.

Physical states also need to satisfy the following condition:

$$L_n|\phi\rangle = 0 \tag{2.35}$$

for $n > 0$, and an identical equation for $\tilde{L}_m$. The algebra generated by these operators is:

$$[L_m, L_n] = (m-n)\, L_{m+n} + \frac{D}{12}\left(m^3 - m\right)\delta_{m,-n} \tag{2.36}$$

where D is the dimension of the space-time.

2.4.1 Derivation of the Virasoro algebra

As we can see from expression eq.(2.36), the commutator of the Virasoro operator consists of two parts. The second term is a quantum anomaly not present in the classical theory.

Notice that the anomalous commutation relations of the Virasoro operators make it impossible to find states annihilated by all of them. A ghost-free spectrum is only possible for certain values of the constant a and the space-time dimension D. If there are no ghosts among the allowed states, then Lorentz invariance of the covariant formalism ensures that there are no ghosts in the physical Hilbert space, which only makes physical sense when the space-time has the critical dimension of 26. In 26 dimensions there are no negative norm physical states.

In order to appreciate the statements above let us derive them explicitly. First let us obtain the classic commutator relations for the Virasoro operators.

From the definition of the Virasoro operator we have:

$$[L_m, L_n] = \frac{1}{4}\sum_{k,l=-\infty}^{\infty} \left[\alpha_{m-k} \cdot \alpha_k, \alpha_{n-l} \cdot \alpha_l\right], \tag{2.37}$$

where the dot means a scalar product. We can then use the identity $[AB, CD] = A[B, C]D + AC[B, D] + [A, C]DB + C[A, D]B$. So the commutator becomes:

$$
\begin{aligned}
[L_m, L_n] &= \frac{1}{4}\sum_{k,l=-\infty}^{\infty}\left(k\alpha_{m-k}\cdot\alpha_l\delta_{k+n-l} + k\alpha_{m-k}\cdot\alpha_{n-l}\delta_{k+l}\right.\\
&\quad\left. + (m-k)\,\alpha_l\cdot\alpha_k\delta_{m-k+n-l} + (m-k)\,\alpha_{n-l}\cdot\alpha_k\delta_{m-k+l}\right). \qquad (2.38)
\end{aligned}
$$

We can simplify the expression above obtaining the following:

$$
[L_m, L_n] = \frac{1}{2}\sum_{k=-\infty}^{\infty}\left(k\alpha_{m-k}\cdot\alpha_{k+n} + (m-k)\,\alpha_{m-k+n}\cdot\alpha_k\right). \qquad (2.39)
$$

Changing variables in the first term ($k \to k' = k+n$), we arrive at the classical version of the Virasoro algebra

$$
[L_m, L_n] = (m-n)\,L_{m+n}. \qquad (2.40)
$$

Let us proceed now to compute the quantum counterpart of the Virasoro algebra [75, 78, 79]. We start from

$$
L_m = -\frac{1}{4\pi}\int_{-\pi}^{\pi} d\sigma e^{im\sigma}\left(\dot{X} + X'\right)^2.
$$

Making the following change of variable: $z = e^{i\sigma}$ we have

$$
L_m = i\frac{1}{4\pi}\int_{c(z)} dz z^{m-1}\mathcal{P}^\mu(z)\mathcal{P}_\mu(z), \qquad (2.41)
$$

where

$$
\mathcal{P}^\mu(z) = \dot{X}^\mu + X'^\mu. \qquad (2.42)
$$

From here we see that

$$
\begin{aligned}
[L_m, L_n] &= -\frac{1}{16\pi^2}\int_{c(z)}\int_{c(w)} dz\,dw\,z^{m-1}w^{n-1}\mathcal{P}^\mu(z)\mathcal{P}_\mu(z)\mathcal{P}^\nu(w)\mathcal{P}_\nu(w)\\
&\quad+\frac{1}{16\pi^2}\int_{c(z)}\int_{c(w)} dz\,dw\,z^{m-1}w^{n-1}\mathcal{P}^\mu(w)\mathcal{P}_\mu(w)\mathcal{P}^\nu(z)\mathcal{P}_\nu(z). \qquad (2.43)
\end{aligned}
$$

Time ordering this expression, we obtain:

$$
\begin{aligned}
[L_m, L_n] &= -\frac{1}{16\pi^2}\int_{c(z)}\int_{c(w)} z^{m-1}w^{n-1}\left[:\mathcal{P}^\mu(z)\mathcal{P}_\mu(z)\mathcal{P}^\nu(w)\mathcal{P}_\nu(w):\right.\\
&\quad\left.+4\underline{\mathcal{P}^\mu(z)\mathcal{P}^\nu(w)}:\mathcal{P}_\mu(z)\mathcal{P}_\nu(w): +2\underline{\mathcal{P}^\mu(z)\mathcal{P}^\nu(w)}\,\underline{\mathcal{P}_\mu(z)\mathcal{P}_\nu(w)}\right] +\\
&\quad\frac{1}{16\pi^2}\int_{c(z)}\int_{c(w)} z^{m-1}w^{n-1}\left[:\mathcal{P}^\mu(w)\mathcal{P}_\mu(w)\mathcal{P}^\nu(z)\mathcal{P}_\nu(z):\right.\\
&\quad\left.+4\underline{\mathcal{P}^\mu(w)\mathcal{P}^\nu(z)}:\mathcal{P}_\mu(w)\mathcal{P}_\nu(z): +2\underline{\mathcal{P}^\mu(w)\mathcal{P}^\nu(z)}\,\underline{\mathcal{P}_\mu(w)\mathcal{P}_\nu(z)}\right] \qquad (2.44)
\end{aligned}
$$

and :: means normal order and $\underline{\quad\quad}$ means a Dyson-Wick contraction [80]:

$$
\underline{AB} = AB - :AB:.
$$

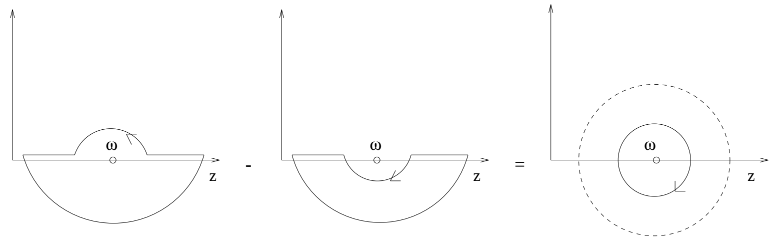

Figure 2.2: The z-contour for each w.

Using the explicit form of $\mathcal{P}^\mu(z)$ (2.42) we find that

$$\underline{\mathcal{P}^\mu(z)\mathcal{P}^\nu(w)} = 2\sum_{m=1}^{\infty} m\eta^{\mu\nu}\left(\frac{w}{z}\right)^m, \tag{2.45}$$

if and only if $w < z$. Substituting eq.(2.45) back into expression (2.44) we obtain

$$\begin{aligned}
[L_m, L_n] \quad = \quad & -\frac{1}{16\pi^2}\int_{c(z)}\int_{c(w)} dz\, dw\; z^{m-1}w^{n-1}\,[:\mathcal{P}^\mu(z)\mathcal{P}_\mu(z)\mathcal{P}^\nu(w)\mathcal{P}_\nu(w): \\
& + \frac{8wz}{(z-w)^2} : \mathcal{P}^\nu(z)\mathcal{P}_\nu(w) : + \frac{8w^2z^2}{(z-w)^4}D\Big] \quad w < z \\
& + \frac{1}{16\pi^2}\int_{c(z)}\int_{c(w)} dz\, dw\; z^{m-1}w^{n-1}\,[:\mathcal{P}^\mu(w)\mathcal{P}_\mu(w)\mathcal{P}^\nu(z)\mathcal{P}_\nu(z): \\
& + \frac{8wz}{(z-w)^2} : \mathcal{P}^\nu(w)\mathcal{P}_\nu(z) : + \frac{8w^2z^2}{(z-w)^4}D\Big] \quad z < w.
\end{aligned} \tag{2.46}$$

Using the contours of fig.(2.2) we obtain the following results: the first integral in eq.(2.46) identically vanishes. For the second integral we have:

$$\begin{aligned}
\int_{c(z)}\int_{c(w)} dz\, dw\, \frac{z^m w^n}{(z-w)^2} : \mathcal{P}^\nu(z)\mathcal{P}_\nu(w) : \quad = \quad & 2\pi i\, m\int_{c(w)} dw\frac{w^{n+m}}{w} : \mathcal{P}^\nu(w)\mathcal{P}_\nu(w) : \\
& + 2\pi i\int_{c(w)} dw\; w^{m+n} : \mathcal{P}'^\nu(w)\mathcal{P}_\nu(w) :
\end{aligned} \tag{2.47}$$

where $\mathcal{P}'^\nu(w) = d\mathcal{P}(w)/dw$. The second integral in (2.47) can be written as follows:

$$\begin{aligned}
2\pi i\int_{c(w)} dw\; w^{m+n} : \mathcal{P}'^\nu(w)\mathcal{P}_\nu(w) : \quad = \quad & \pi\, i\int_{c(w)} dw\frac{d}{dw}\left(w^{n+m} : \mathcal{P}^\nu(w)\mathcal{P}_\nu(w) :\right) \\
& - \pi\, i\,(n+m)\int_{c(w)} dw\frac{w^{n+m}}{w} : \mathcal{P}^\nu(w)\mathcal{P}_\nu(w) :
\end{aligned} \tag{2.48}$$

and from this expression we obtain:

$$2\pi i\int_{c(w)} dw\; w^{m+n} : \mathcal{P}'^\nu(w)\mathcal{P}_\nu(w) : \quad = \quad 4\,(m+n)\,L_{m+n}. \tag{2.49}$$

Looking now at the first integral in eq.(2.47) we see that

$$2\pi i\ m \int_{c(w)} dw \frac{w^{n+m}}{w} : \mathcal{P}^{\nu}(w)\mathcal{P}_{\nu}(w) := -8mL_{m+n}. \tag{2.50}$$

Finally, the third integral in eq.(2.46) gives:

$$\int_{c(z)}\int_{c(w)} dz\ dw\ \frac{z^{m+1}w^{n+1}}{(z-w)^4} D = -\frac{2\pi^2}{3} mD \left(m^2-1\right) \delta_{m,-n}. \tag{2.51}$$

Substituting eqs.(2.49)-(2.51) back into eq.(2.46), we obtain our final result:

$$[L_m, L_n] = (m-n)\, L_{m+n} + \frac{D}{12}\left(m^3 - m\right)\delta_{m,-n} \tag{2.52}$$

Now, let us derive the critical dimension of the space-time and the constant a appearing in eq.(2.28). We need to vary the parameters a and D in order to find the regions where there are no negative norm states in the physical Hilbert space. For simplicity, let us work with the open string case. If we denote the ground state of the open string with momentum k^{μ} as $|0;k\rangle$, we find that the mass-shell condition $L_0 = a$ means that $k^2 = -2a$. If we go to the first excited level, we will need to introduce a polarisation vector with D independent components (before gauging away some of them). The states in the first excitation level are given by

$$\zeta \cdot \alpha_{-1}|0;k\rangle.$$

The mass-shell condition now implies $k^2 = -2(a-1)$, and the L_1 subsidiary condition implies that $\zeta \cdot k = 0$. This means that we only have $D-1$ allowed polarisations. The norm of these states is given by $\zeta\cdot\zeta$. If the vector k^{μ} lies in the $(0,\vec{1})$ plane, then the $D-2$ states with polarisation normal to that plane have positive norms. If the first excited state is a tachyon, then $k^2 < 0$ and k^{μ} can be chosen to have no time component. This state has a negative norm. If $k^2 > 0$, k can be chosen to have only a time component and the norm will be positive. Finally if $k^2 = 0$, we find that the norm is zero. With these results we find that one of the conditions for the absence of unphysical states is

$$a \leq 1.$$

Now let us define a *spurious* state. A spurious state satisfies the following conditions:

$$(L_0 - a)\,|\psi\rangle = 0$$

and

$$\langle\phi|\psi\rangle = 0,$$

for all physical states $|\phi\rangle$. These states can be written in the form

$$|\psi\rangle = \sum_{n>0} L_{-n}|\chi_n\rangle,$$

where $|\chi_n\rangle$ is a state that satisfies

$$(L_0 - a + n)\,|\chi_n\rangle = 0.$$

Now, let us now construct a state that is both spurious and physical. With this in mind, let us define a spurious state of the following form:

$$|\psi\rangle = L_{-1}|\chi\rangle,$$

where χ is an arbitrary state which satisfies: $L_m|\chi\rangle = 0$ for $m > 0$ and $(L_0 - a + 1)\,|\chi\rangle = 0$. The state $|\psi\rangle$ is not a physical state as it stands. Let us apply the operator L_1 to this state thus:

$$L_1|\psi\rangle = L_1 L_{-1}|\chi\rangle = 2L_0|\chi\rangle,$$

which does not vanish unless $a = 1$. Now let us construct another spurious state as follows:

$$|\psi\rangle = \left(L_2 + \gamma L_{-1}^2\right)|\chi\rangle.$$

This state needs to satisfy $L_1|\psi\rangle = L_2|\psi\rangle = 0$ in order to be a physical state as well. From these conditions we find two equations:

$$\left(L_1 L_{-2} + \gamma L_1 L_{-1}^2\right)|\psi\rangle = 0$$

and

$$\left(L_2 L_{-2} + \gamma L_2 L_{-1}^2\right)|\psi\rangle = 0.$$

From these equations we find now that $\gamma = 3/2$ and that $D = 26$. Notice that from this discussion about spurious states we have found a set of states which are physical and whose norm is zero. That is, they are at the *border* between states with negative norm and states with positive norm. Therefore, we have found that $D = 26$ is a critical dimension. Of course, this derivation of the critical dimension D of the space-time and of the value of the parameter a is far from rigourous; however, it gives a physical insight as to how they emerge in String Theory. It can be proved that the bosonic string spectrum has no ghosts and the theory is unitary for precisely these values of the space-time dimension D and the parameter a [76].

2.5 Vertex operators

Interactions can be seen in the open String Theory as a process in which a single string splits into two or one in which two strings join together to form a single one (see for example fig.(2.3)). In String Theory one is naturally led to introduce an operator by means of which a second string can be obtained from the original string. This operator, in the open string case, is introduced at the ends of the string where string 2 is emitted.

2.5.1 Conformal dimension

Let us start by considering open strings only. Let us now introduce a local operator $A(\tau)$. This operator is in reality a function of both σ and τ but because we want to study this operator at the end-points of the string [76] (we are working with open strings) we have chosen $\sigma = 0$ for simplicity, although we could have chosen the other end-point: $\sigma = 2\pi$. $A(\tau)$ is defined to have conformal dimension J **if and only if** under reparametrization it transforms to

$$A'(\tau') = \left(\frac{d\tau}{d\tau'}\right)^J A(\tau).$$

If we now consider an infinitesimal transformation: $\tau \to \tau' = \tau + \epsilon(\tau)$, then the transformation law of an operator of conformal dimension J reads:

$$\delta A(\tau) = -\epsilon \frac{dA}{d\tau} - JA\frac{d\epsilon}{d\tau}. \tag{2.53}$$

If we look now at the commutation relations between the Virasoro operators and the local operator $A(\tau)$, we find that the condition for $A(\tau)$ to have conformal dimension J is

$$[L_n, A(\tau)] = e^{in\tau}\left(-i\frac{d}{d\tau} + mJ\right) A(\tau). \tag{2.54}$$

If an expansion in Fourier modes of the operator $A(\tau)$ is possible then eq.(2.54) becomes:

$$[L_n, A_m] = [n\,(J-1) - m]\, A_{n+m}. \tag{2.55}$$

Using this expression and after some algebra we find that the string coordinate $X^\mu(\tau)$ has conformal dimension $J = 0$ and $\dot{X}^\mu(\tau)$ has conformal dimension $J = 1$. The operators that satisfy eq.(2.54) are said to have a *definite* conformal dimension. These operators are useful because with them we can build physical states from others. For example, if we have a physical state $|\phi\rangle$ and $A(\tau)$ has conformal dimension $J = 1$, we can easily see that

$$[L_n, A_0] = 0$$

and therefore that

$$|\phi'\rangle = A_0|\phi\rangle$$

is also a physical state.

Looking back to the issue of constructing a vertex operator, we see that vertex operators need to map an initial physical state to a final physical state. From the discussion above it is expected that the open string operator should have conformal dimension $J = 1$ ($J = 2$ in the case of closed strings). Vertex operators need to be conformally invariant in order for the theory to make sense (to be anomaly free), which means that it needs to be invariant under reparametrization of the world-sheet. The vertex operator $V(k, \tau)$ for the emission at time τ and $\sigma = 0$ (or π) of a physical state of momentum k^μ or absorption of a physical state of momentum $-k^\mu$ has the following general structure:

$$V = \int d\tau \, W_v(X^\mu, \dot{X}^\mu, \ddot{X}, \ldots)e^{ik\cdot X(0,\tau)} \tag{2.56}$$

for open strings, and

$$V = \int d\sigma d\tau \, W_v(X^\mu, \dot{X}^\mu, \ddot{X}, \ldots)e^{ik\cdot X(\sigma,\tau)} \tag{2.57}$$

for closed strings.

It must change, amongst several other things, the momentum of the states it acts on by an amount k^μ. This is achieved by the introduction of the factors $e^{ik\cdot X(0,\tau)}$ and $e^{ik\cdot X(\sigma,\tau)}$ which appear in the vertex operator for open and closed strings respectively. Notice that these factors by themselves require normal ordering.

We can see that the main difference between open and closed string vertex operators is that in the closed String Theory the vertex operator is introduced in the world-sheet instead of the ends of the string. Another difference is that the conformal dimension of the vertex operator differs by a factor of 2 from that for the open string.

2.6 The bosonic string spectrum

It was in the early 1960's that it was realised that hadronic resonances possessed rather high spins which increased proportionally to the mass squared of the state following roughly the prescription: $m^2 = J/\alpha'$ where J is the spin of the state, m the mass of the state and α' again the Regge slope. If we plot this with the energy squared on the $x-$axis and the spin of the state on the $y-$axis, then we obtain a curve called the *Regge*

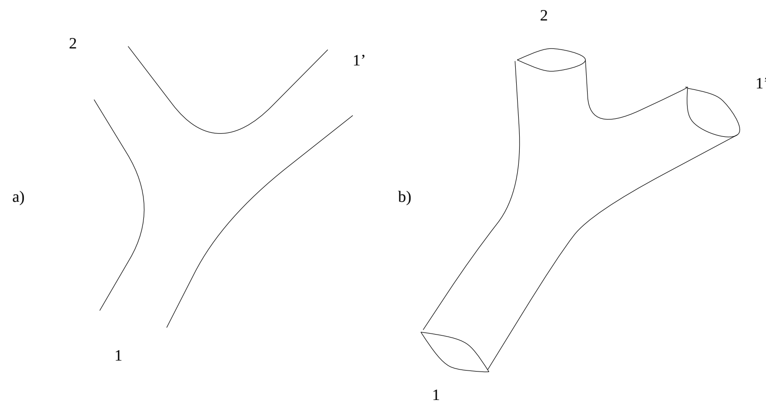

Figure 2.3: a) Splitting of open strings and b) closed strings.

trajectory. As is well known today, String Theory started as an attempt to explain the physics of the strong interaction including this behaviour of the mass spectrum. Indeed the theory possesses a similar behaviour for the mass spectrum of its states. At that time the setbacks of the theory (as a theory for strong interactions) were (amongst others) that the string spectrum also has massless states which do not belong to the hadronic world of strong interactions and it has the wrong Regge slope. However, nowadays we know that String Theory is not the correct theory for studying strong interactions but rather a theory which may provide us with a unified picture of all known interactions. Having said this, let us see what the string spectrum looks like. The spectrum of the lower-lying states in the open string case can be categorised as (see fig.2.4):

a) Tachyon$\rightarrow |0\rangle$

b) Massless vector$\rightarrow a_1^{\dagger\mu}|0\rangle$

c) Massless scalar$\rightarrow k_\mu a_1^{\dagger\mu}|0\rangle$

d) Massive spin two$\rightarrow a_1^{\dagger\mu}a_1^{\dagger\nu}|0\rangle$

e) Massive vector$\rightarrow a_2^{\dagger\mu}|0\rangle$

and for closed string it can be categorised as[2] (see fig.2.5)

a') Tachyon$\rightarrow |0\rangle$

[2]The masses of these particles may be obtained quickly from the conformal dimension of the appropriate vertex operator as discussed in the previous section. In short, knowing that the vertex operators for open string states have to have conformal dimension $J=1$ and the ones for closed string states have to have conformal dimension $J=2$, we can calculate the mass squared of the state in question: $J_v=-k^2/2$ for open strings and $J_v=-k^2/4$ for closed strings. Here J_v is the conformal dimension of the factor $e^{ik\cdot X}$ appearing in all vertex operators of String Theory.

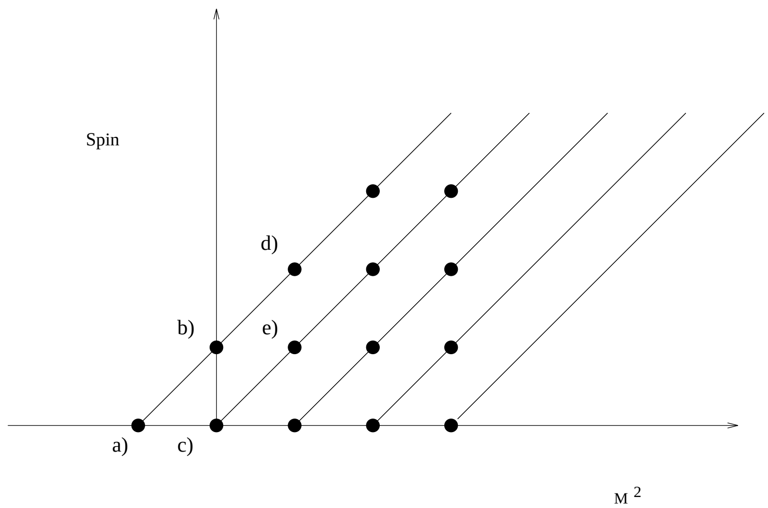

Figure 2.4: Regge trajectories for open strings.

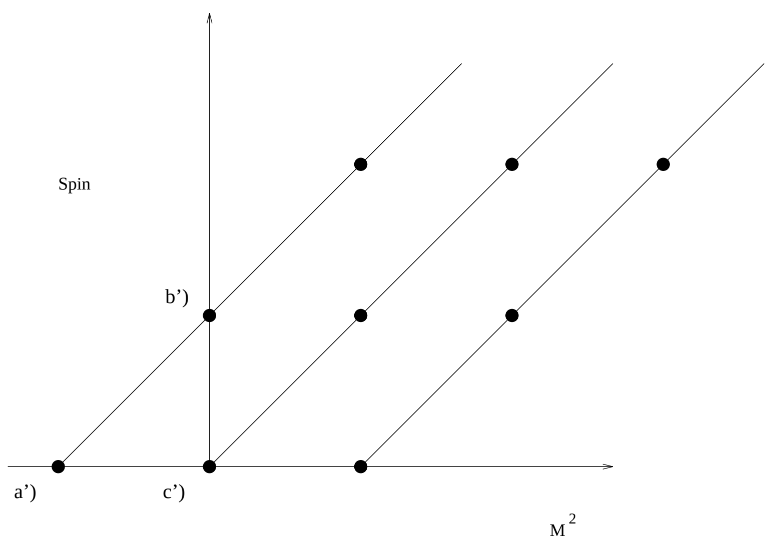

Figure 2.5: Regge trajectories for closed strings.

b') Massless spin two→ $a_1^{\dagger\mu} a_1^{\dagger\nu}|0\rangle$

c') Massless scalar→ $k_\mu k_\nu a_1^{\dagger\mu} \tilde{a}_1^{\dagger\nu}|0\rangle$

It is the massless spin two particle appearing in the mass spectrum of the closed string that made people think of String Theory as a possible theory of quantum gravity. This particle is the graviton. As we can see, the graviton enters the theory as a member of a multiplet. The other important massless particle appearing in the spectrum for closed string is the scalar massless particle. This particle is the dilaton.

So we can see that the description of free bosonic strings gives a theory which is much more constrained than any other corresponding theory for point-particles.

2.7 Fixing the gauge

There are three formalisms in which to fix the gauge of the theory. This means that there exist at least three ways of first quantising the string [77]:

1. The Gupta-Bleuler is perhaps the simplest of the three formalisms. We allow ghosts to appear in the action, which permits us to maintain manifest Lorentz invariance. The price we must pay, however, is that we must impose ghost-killing constraints on the Hilbert space. Projection operators must be inserted in all propagators. At tree level, but for higher loops, this is extremely difficult.

   The Gupta-Bleuler formalism will maintain Lorentz invariance by imposing the Virasoro constraints on the state vectors of the theory:

   $$\langle\phi|L_f|\psi\rangle = 0 \tag{2.58}$$

   where $\langle\phi|$ and $|\psi\rangle$ represent states of the theory. These constraints will eliminate the ghost which appears in the state vectors. Therefore, we are allowed to maintain the non-physical states of the theory in the action.

2. The light cone gauge formalism has the advantage that it is explicitly ghost-free in the action as well as in the Hilbert space. There exist no complications when we go to higher loops. However, the formalism is slightly awkward and we have to check Lorentz invariance at each step.

3. The BRST formalism combines the best features of the two formalisms mentioned above. It is manifestly covariant, like the Gupta-Bleuler formalism, and it is unitary, like the light cone formalism. This is because the negative norm ghosts cancel against the Faddeev-Popov ghosts which need to be introduced in the string action.

We will be using mainly the Gupta-Bleuler formalism. Notice that most of the results presented in this chapter were derived in this formalism.

2.8 String compactification

One of the most important problems of String Theory is the one regarding the extra dimensions of the theory. As we have seen, bosonic string theory is only consistent in a 26-dimensional space-time whereas the more realistic superstring counterpart only makes

sense in a 10-dimensional space-time. Therefore, we need somehow to compactify the theory to a realistic 4-dimensional one (see for example [49] - [57]). Until such a reduction is made, the theory will have a lack of contact with real physical observables.

Because we do not have a quantum field theory framework on which this dimensional breaking can possibly be done, the best we can do is to look at classical solutions where spontaneous compactification of the extra dimensions has already taken place.

CHAPTER 3

The quantum bosonic string energy-momentum tensor in Minkowski space-time

In this chapter[1] and the next, I will present some of the main derivations and results of this thesis. Here we will compute the quantum energy-momentum tensor $\hat{T}^{\mu\nu}(x)$ for bosonic strings in Minkowski space-time [96]. Its expectation value, for different physical string states both for open and closed bosonic strings, will be computed. The states considered are described by normalizable wave-packets in the centre of mass coordinates. We will find in particular that $\hat{T}^{\mu\nu}(x)$ loses its locality, which could imply that the classical divergence that occurs in String Theory as we approach the string position is removed at the quantum level as the string position is smeared out by quantum fluctuations.

3.1 Introduction

String Theory has emerged as the most promising candidate to reconcile general relativity with quantum mechanics and unify gravity with the other fundamental interactions. Because of this merging of quantum field theory with general relativity in String Theory, it makes sense to investigate the gravitational consequences of strings as we approach the Planck scale. We must remember that when particles scatter at energies of the order of or larger than the Planck mass, the interaction that dominates their collision is the gravitational one. At these energies the picture of particle fields or strings in flat space-time ceases to be valid, the curved space-time geometry created by the particles has to be taken into account. This has been the motivation here, to investigate the possible gravitational effects arising from an isolated quantum bosonic string living in a flat space-time background, so we may begin a study of the scattering process of strings merely by the gravitational interaction between them. A systematic and thorough study of quantum strings in physically relevant curved space-times has been started in [82] and has been reviewed in ref. [83].

[1]This chapter is based upon work with E. J. Copeland and H. J. de Vega.

As we have said, in this chapter, we calculate the energy-momentum tensor of both closed and open quantum bosonic strings in $3+1$ dimensions. Our target space is assumed to be the direct product of four dimensional Minkowski space-time times a compact manifold taking care of conformal anomalies.

As a starting point we recall that it has already been shown [85] that the back-reaction for a classical bosonic string in $3+1$ dimensions has a logarithmic divergence when the space-time coordinate $x \to X(\sigma)$; that is, when we approach the core of the string. This divergence is absorbed into a renormalization of the string tension. Copeland et al [85] showed that by demanding that both it and the divergence in the energy-momentum tensor vanish, the string is forced to have the couplings of compactified $N = 1$, $D = 10$ supergravity. In this calculation we will be able to see that when we take into account the quantum nature of the strings, we lose all information regarding the position of the string and therefore any divergences that may appear when one calculates the back-reaction of quantum strings are not related at all to the classical position of the string.

In the present work, the energy-momentum tensor of the string, $\hat{T}^{\mu\nu}$, is a quantum operator and it may be regarded as a vertex operator for the emission (absorption) of gravitons. We compute its expectation value in one-particle string states, choosing for the string centre of mass wave function a wave-packet centred at the origin. Notice that this computation will preserve all the *stringy* features, in contrast to similar computations presented in [10], where these features are lost since they integrate the energy-momentum tensor of the bosonic string over a spatial volume totally enclosing the string.

Our results will depend on the mass of the string state chosen. We will concentrate mainly on the string massless states although some results for the string tachyonic state will also be shown. In our computation, we consider spherically symmetric and cylindrically symmetric configurations for our string states. The components for the string energy density and energy flux behave like massless waves, with the string energy being radiated outwards as a massless lump peaked at $r = t$ (in the spherically symmetric case). We provide integral representations for $\langle \hat{T}^{\mu\nu}(r,t) \rangle$ [eq.(3.51)]. We can see that $\langle \hat{T}^{\mu\nu}(r,t) \rangle$ propagates as outgoing (plus ingoing) spherical waves. After exhibiting the tensor structure of $\langle \hat{T}^{\mu\nu} \rangle$ [eqs.(3.53)-(3.56)], the asymptotic behaviour of $\langle \hat{T}^{\mu\nu}(r,t) \rangle$ for $r \to \infty$ and t fixed and for $t \to \infty$ with r fixed is computed. In the first regime the energy density and the stress tensor decay as r^{-1} whereas the energy flux decays as r^{-2}. For $t \to \infty$ with r fixed, the energy density tends to 0^- as e^{-t^2}. That is, the spherical wave leaves behind a

rapidly vanishing negative energy density.

For cylindrically symmetric configurations, $\langle \hat{T}^{\mu\nu}(\rho,t)\rangle$ propagates as outgoing (plus ingoing) cylindrical waves. For large ρ and fixed t the energy density decays as $1/\rho$ and the energy flux decays as $1/\rho^2$. For large $t \to \infty$ with ρ fixed.

3.2 The string energy-momentum tensor

We recall that the energy-momentum tensor for a classical bosonic string with tension $(\alpha')^{-1}$ (see for example [10]) is given by

$$T^{\mu\nu}(x) = \frac{1}{2\pi\alpha'}\int d\sigma d\tau \; (\dot{X}^\mu \dot{X}^\nu - X'^\mu X'^\nu)\delta(x - X(\sigma,\tau)) \tag{3.1}$$

This expression is easily obtained from the Polyakov form of the string action presented in the previous chapter by varying the action with respect to the space-time metric. The delta function in the expression above comes from the fact that the string energy-momentum tensor needs to vanish everywhere except at the position of the string. Jumping a few steps ahead, we notice that at the quantum level the locality given by this delta function will disappear since quantum fluctuations will smear the localised behaviour of the classical string.

The string coordinates are given in Minkowski space-time by

$$X^\mu(\sigma,\tau) = q^\mu + 2\alpha' p^\mu \tau + i\sqrt{\alpha'}\sum_{n\neq 0}\frac{1}{n}[\alpha_n^\mu \; e^{-in(\tau-\sigma)} + \tilde{\alpha}_n^\mu \; e^{-in(\tau+\sigma)}] \tag{3.2}$$

for closed strings and

$$X^\mu(\sigma,\tau) = q^\mu + 2\alpha' p^\mu \tau + i\sqrt{\alpha'}\sum_{n\neq 0}\frac{1}{n}\; \alpha_n^\mu \; e^{-in\tau}\cos n\sigma \tag{3.3}$$

for open strings, as was also stated in the previous chapter.

For closed strings we can set $\alpha' = 1/2$; thus, inserting eq.(3.2) in eq.(3.1) and rewriting the four dimensional delta function in integral form, we obtain for closed strings:

$$\begin{aligned} T^{\mu\nu}(x) = \frac{1}{\pi}\int d\sigma d\tau \frac{d^4\lambda}{(2\pi)^4}\{p^\mu p^\nu + \frac{p^\mu}{\sqrt{2}}\sum_{n\neq 0}[\alpha_n^\nu e^{-in(\tau-\sigma)} + \tilde{\alpha}_n^\nu e^{-in(\tau+\sigma)}] + \\ + \sum_{n\neq 0}(\alpha_n^\mu e^{-in(\tau-\sigma)} + \tilde{\alpha}_n^\mu e^{-in(\tau+\sigma)})\frac{p^\nu}{\sqrt{2}} + \\ + \sum_{n\neq 0}\sum_{m\neq 0}[\alpha_n^\mu \tilde{\alpha}_m^\nu e^{-in(\tau-\sigma)}e^{-im(\tau+\sigma)} + \tilde{\alpha}_n^\mu \alpha_m^\nu e^{-in(\tau+\sigma)}e^{-im(\tau-\sigma)}]\}\; e^{i\lambda\cdot x}\; e^{-i\lambda\cdot X(\sigma,\tau)}. \end{aligned} \tag{3.4}$$

We can now write

$$X(\sigma,\tau) = X_{cm} + X_+ + X_-,$$

where $X_{cm} = q + p\tau$ is the centre of mass coordinate and X_+ and X_- refer to the terms with $\alpha_{n>0}$ and $\alpha_{n<0}$ in $X(\sigma,\tau)$ respectively. In this way we can now see that our energy-momentum tensor has the same form as that of a vertex operator [76, 77].

We should notice that eq.(3.1) is meaningful at the classical level. However, at the quantum level one must be careful with the order of the operators since $\dot{X}^\mu$ and $\dot{X}^\nu$ do not commute with $X(\sigma,\tau)$. We shall define the quantum operator $\hat{T}^{\mu\nu}(x)$ by symmetric ordering. That is,

$$\hat{T}^{\mu\nu}(x) \equiv \frac{1}{3}\left[\hat{T}_a^{\mu\nu}(x) + \hat{T}_b^{\mu\nu}(x) + \hat{T}_c^{\mu\nu}(x)\right] \tag{3.5}$$

where

$$\begin{aligned}
\hat{T}_a^{\mu\nu}(x) &= \frac{1}{2\pi\alpha'}\int d\sigma d\tau \left(\dot{X}^\mu\dot{X}^\nu - X'^\mu X'^\nu\right)\delta(x - X(\sigma,\tau)) \\
\hat{T}_b^{\mu\nu}(x) &= \frac{1}{2\pi\alpha'}\int d\sigma d\tau \delta(x - X(\sigma,\tau))\left(\dot{X}^\mu\dot{X}^\nu - X'^\mu X'^\nu\right) \\
\hat{T}_c^{\mu\nu}(x) &= \frac{1}{2\pi\alpha'}\int d\sigma d\tau \left[\frac{1}{2}\left(\dot{X}^\mu\delta(x - X(\sigma,\tau))\dot{X}^\nu - X'^\mu\delta(x - X(\sigma,\tau))X'^\nu + \right.\right. \\
&\quad \left.\left.\dot{X}^\nu\delta(x - X(\sigma,\tau))\dot{X}^\mu - X'^\nu\delta(x - X(\sigma,\tau))X'^\mu\right)\right].
\end{aligned} \tag{3.6}$$

This definition ensures hermiticity:

$$\hat{T}^{\mu\nu}(x)^\dagger = \hat{T}^{\mu\nu}(x).$$

Since we are interested in computing an expectation for the above quantum operator, we must use invariantly normalizable particle states as wave packets.

Let us consider a string on a mass and spin eigenstate with a centre of mass wave function. An on-shell scalar string state is then

$$|\Psi\rangle = \int d^4p\, \frac{\varphi(\vec{p})}{E}\, \delta(p^0 - \sqrt{\vec{p}^2 + m^2})\, |p\rangle. \tag{3.7}$$

Where

$$E = p^0 = \sqrt{\vec{p}^2 + m^2}$$

and $\varphi(\vec{p})$ is a wave-packet that we are free to choose.

Here we assume the extra space-time dimensions (beyond four) to be appropriately compactified and consider string states in the physical (uncompactified) four dimensional Minkowski space-time.

This string state needs to be correctly normalised. Evaluating the scalar product we obtain:

$$\langle\Psi|\Psi\rangle = \int d^3p d^3p' \delta^3(\vec{p} - \vec{p}')\delta(p'^0 - \sqrt{p'^2 + m^2})\delta(p^0 - \sqrt{p^2 + m^2})\, dp^0 dp'^0 \delta(p^0 - p'^0) \times$$

$$\frac{\varphi^*(\vec{p})}{E}\frac{\varphi(\vec{p}')}{E'} = \int d^3p \frac{|\varphi(\vec{p})|^2}{E^2}\delta(0). \tag{3.8}$$

The last Dirac delta can be regularised by considering a large finite temporal box of size T.

$$\delta(0) = \frac{1}{2\pi}\int_0^T dt = \frac{T}{2\pi}. \tag{3.9}$$

Hence we have

$$\langle\Psi|\Psi\rangle = \frac{T}{2\pi}\int d^3p \frac{|\varphi(\vec{p})|^2}{E^2}. \tag{3.10}$$

Returning to $\hat{T}^{\mu\nu}(x)$, because it is a quantum operator, it requires normal ordering. For closed strings we obtain:

$$\begin{aligned}
\langle\Psi|\hat{T}_a^{\mu\nu}(x)|\Psi\rangle \quad = \quad & \frac{1}{\pi}\int d\sigma d\tau \frac{d^4\lambda}{(2\pi)^4}\, e^{i\lambda\cdot x}\Big\{\langle\Psi|e^{-i\lambda\cdot X_-}p^\mu p^\nu e^{-i\lambda\cdot X_+}e^{-i\lambda\cdot X_{cm}}|\Psi\rangle + \\
& \langle\Psi|e^{-i\lambda\cdot X_-}\frac{p^\mu}{\sqrt{2}}\sum_{n=1}\left[\alpha^\nu_{-n}e^{in(\tau-\sigma)} + \tilde{\alpha}^\nu_{-n}e^{in(\tau+\sigma)}\right]e^{-i\lambda\cdot X_+}e^{-i\lambda\cdot X_{cm}}|\Psi\rangle + \\
& \langle\Psi|e^{-i\lambda\cdot X_-}\sum_{n=1}\left[\alpha^\nu_{n}e^{-in(\tau-\sigma)} + \tilde{\alpha}^\nu_{n}e^{-in(\tau+\sigma)}\right]\frac{p^\mu}{\sqrt{2}}e^{-i\lambda\cdot X_+}e^{-i\lambda\cdot X_{cm}}|\Psi\rangle + \\
& \sum_{n=1}\sum_{m=1}\langle\Psi|e^{-i\lambda\cdot X_-}\alpha^\mu_n\tilde{\alpha}^\nu_m e^{-in(\tau-\sigma)}e^{-im(\tau+\sigma)}e^{-i\lambda\cdot X_+}e^{-i\lambda\cdot X_{cm}}|\Psi\rangle + \\
& \sum_{n=1}\sum_{m=1}\langle\Psi|e^{-i\lambda\cdot X_-}\alpha^\mu_{-n}\tilde{\alpha}^\nu_{-m} e^{in(\tau-\sigma)}e^{im(\tau+\sigma)}e^{-i\lambda\cdot X_+}e^{-i\lambda\cdot X_{cm}}|\Psi\rangle + \\
& \sum_{n=1}\sum_{m=1}\langle\Psi|e^{-i\lambda\cdot X_-}\alpha^\mu_n\tilde{\alpha}^\nu_{-m} e^{-in(\tau-\sigma)}e^{im(\tau+\sigma)}e^{-i\lambda\cdot X_+}e^{-i\lambda\cdot X_{cm}}|\Psi\rangle + \\
& \sum_{n=1}\sum_{m=1}\langle\Psi|e^{-i\lambda\cdot X_-}\alpha^\mu_{-n}\tilde{\alpha}^\nu_m e^{in(\tau-\sigma)}e^{-im(\tau+\sigma)}e^{-i\lambda\cdot X_+}e^{-i\lambda\cdot X_{cm}}|\Psi\rangle + \\
& \sum_{n=1}\sum_{m=1}\langle\Psi|e^{-i\lambda\cdot X_-}\tilde{\alpha}^\mu_n\alpha^\nu_m e^{-in(\tau+\sigma)}e^{-im(\tau-\sigma)}e^{-i\lambda\cdot X_+}e^{-i\lambda\cdot X_{cm}}|\Psi\rangle + \\
& \sum_{n=1}\sum_{m=1}\langle\Psi|e^{-i\lambda\cdot X_-}\tilde{\alpha}^\mu_{-n}\alpha^\nu_{-m} e^{in(\tau+\sigma)}e^{im(\tau-\sigma)}e^{-i\lambda\cdot X_+}e^{-i\lambda\cdot X_{cm}}|\Psi\rangle + \\
& \sum_{n=1}\sum_{m=1}\langle\Psi|e^{-i\lambda\cdot X_-}\tilde{\alpha}^\mu_n\alpha^\nu_{-m} e^{-in(\tau+\sigma)}e^{im(\tau-\sigma)}e^{-i\lambda\cdot X_+}e^{-i\lambda\cdot X_{cm}}|\Psi\rangle + \\
& \sum_{n=1}\sum_{m=1}\langle\Psi|e^{-i\lambda\cdot X_-}\tilde{\alpha}^\mu_{-n}\alpha^\nu_m e^{in(\tau+\sigma)}e^{-im(\tau-\sigma)}e^{-i\lambda\cdot X_+}e^{-i\lambda\cdot X_{cm}}|\Psi\rangle\Big\}
\end{aligned} \tag{3.11}$$

and we obtain similar expressions for the other terms in eq.(3.5). It is clear that in the tachyonic case, since $\hat{T}^{\mu\nu}(x)$ is already normal ordered, $e^{-i\lambda\cdot X_+}$, will annihilate the ground state. Thus, we only need to worry about the action of $e^{-i\lambda\cdot X_{cm}}$ on the state $|p_2\rangle$ (Here, $|p_2\rangle$ is the initial momenta and $|p_1\rangle$ is the final momenta). We obtain the following result:

$$\begin{aligned}
&\frac{\langle\Psi|\hat{T}^{\mu\nu}(x)|\Psi\rangle}{\langle\Psi|\Psi\rangle} \equiv \langle\hat{T}^{\mu\nu}(x)\rangle = \\
&\frac{1}{3\pi\langle\Psi|\Psi\rangle}\int \frac{d^4\lambda}{(2\pi)^4}\, d^4p_1 d^4p_2\, d\sigma d\tau\, e^{i\lambda\cdot x}[p_1^\mu p_1^\nu + p_2^\mu p_2^\nu + \frac{1}{2}\left(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu\right)]\times \\
&\langle p_1|e^{-i\lambda\cdot X_{cm}}|p_2\rangle\, \frac{\varphi^*(\vec{p}_1)}{E_1}\frac{\varphi(\vec{p}_2)}{E_2}\, \delta(p_1^0 - \sqrt{\vec{p}_1^2 + m_1^2})\delta(p_2^0 - \sqrt{\vec{p}_2^2 + m_2^2}).
\end{aligned} \tag{3.12}$$

Writing

$$\langle p_1|e^{-i\lambda\cdot X_{cm}}|p_2\rangle = e^{i\frac{\tau\lambda^2}{2}-i\lambda\cdot p_2\tau}\delta^4(\lambda+p_1-p_2), \tag{3.13}$$

eq.(3.12) becomes

$$\langle\hat{T}^{\mu\nu}(x)\rangle = \frac{2}{3\langle\Psi|\Psi\rangle}\int\frac{d^4\lambda}{(2\pi)^3}\,d^4p_1d^4p_2\,d\tau\,e^{i\lambda\cdot x}\,[p_1^\mu p_1^\nu+p_2^\mu p_2^\nu+\frac{1}{2}\,(p_1^\mu p_2^\nu+p_1^\nu p_2^\mu)]\times$$
$$e^{i\frac{\tau\lambda^2}{2}-i\lambda p_2\tau}\,\delta^4(\lambda+p_1-p_2)\frac{\varphi^*(\vec{p}_1)}{E_1}\frac{\varphi(\vec{p}_2)}{E_2}\,\delta(p_1^0-\sqrt{\vec{p}_1^2+m_1^2})\delta(p_2^0-\sqrt{\vec{p}_2^2+m_2^2})\,. \tag{3.14}$$

Performing the λ, p_1^0 and p_2^0 integrals, we get:

$$\langle\hat{T}^{\mu\nu}(x)\rangle = \frac{4}{3(2\pi)^3\langle\Psi|\Psi\rangle}\int d^3p_1d^3p_2\,e^{i(p_2-p_1)\cdot x}\,[p_1^\mu p_1^\nu+p_2^\mu p_2^\nu+\frac{1}{2}\,(p_1^\mu p_2^\nu+p_1^\nu p_2^\mu)]\times$$
$$\frac{\varphi^*(\vec{p}_1)}{E_1}\frac{\varphi(\vec{p}_2)}{E_2}\int_{\tau_2}^{\tau_1}d\tau\,e^{i\frac{\tau}{2}(p_1^2-p_2^2)}. \tag{3.15}$$

The calculation for open strings is obtained by substituting eq.(3.3) into eq.(3.5), with the result (setting $\alpha'=1$):

$$\langle\hat{T}^{\mu\nu}(x)\rangle = \frac{2}{3(2\pi)^3\langle\Psi|\Psi\rangle}\int d^3p_1d^3p_2\,e^{i(p_2-p_1)\cdot x}\,[p_1^\mu p_1^\nu+p_2^\mu p_2^\nu+\frac{1}{2}\,(p_1^\mu p_2^\nu+p_1^\nu p_2^\mu)]\times$$
$$\frac{\varphi^*(\vec{p}_1)}{E_1}\frac{\varphi(\vec{p}_2)}{E_2}\int_{\tau_2}^{\tau_1}d\tau\,e^{i\frac{\tau}{2}(p_1^2-p_2^2)}. \tag{3.16}$$

The limits in the τ integral, come from some very early time τ_2 to some much later time τ_1. We are thinking here of state two as the *in state* and state one as the *out state*.

Now, we know from the fact we are on shell (i.e. the p^0 integrals we have performed) that $p_1^2=p_2^2=m^2$. Hence the *tau* integral becomes:

$$\int_{\tau_2}^{\tau_1}d\tau = \tau_1-\tau_2.$$

This time is not the physical time T and we would like to write it in terms of the physical time. We can do this from the earlier expression for the coordinate of the string. We expect at very early and very late times that the asymptotic behaviour of the string to be dominated by the zero modes. In particular for the physical time coordinate X^0 we can write it as

$$X^0 = q^0 + 2\alpha' p^0\tau,$$

a result true in the limits $\tau\to\pm\infty$, $X^0\to\pm\infty$. Let us call the initial physical time $-T/2$ and the final physical time $+T/2$. Then we have the following two equations:

$$-T/2 = q^0 + 2\alpha' E_2\tau_2$$

and

$$T/2 = q^0 + 2\alpha' E_1 \tau_1$$

Since $\tau_1, \tau_2 \to \infty$ we can ignore the q^0 terms in the expressions above. Taking the difference we reach the following result:

$$\tau_1 - \tau_2 = \frac{T}{4\alpha'}\left(\frac{1}{E_1} + \frac{1}{E_2}\right).$$

We can now replace τ terms in the expressions for $\langle \hat{T}^{\mu\nu}(x)\rangle$ to obtain:

$$\langle \hat{T}^{\mu\nu}(x)\rangle = \frac{2T}{3(2\pi)^3\langle\Psi|\Psi\rangle}\int d^3p_1 d^3p_2\; e^{i(p_2-p_1)\cdot x}\; [p_1^\mu p_1^\nu + p_2^\mu p_2^\nu + \frac{1}{2}\,(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu)] \times \varphi^*(\vec{p}_1)\varphi(\vec{p}_2)\;\left(\frac{E_1+E_2}{(E_1E_2)^2}\right) \tag{3.17}$$

for closed strings, whereas for open strings we obtain

$$\langle \hat{T}^{\mu\nu}(x)\rangle = \frac{T}{6(2\pi)^3\langle\Psi|\Psi\rangle}\int d^3p_1 d^3p_2\; e^{i(p_2-p_1)\cdot x}\; [p_1^\mu p_1^\nu + p_2^\mu p_2^\nu + \frac{1}{2}\,(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu)] \times \varphi^*(\vec{p}_1)\varphi(\vec{p}_2)\;\left(\frac{E_1+E_2}{(E_1E_2)^2}\right). \tag{3.18}$$

Now, let us consider the massless string states. For the closed string the massless states are the graviton

$$|p,1\rangle_s \equiv P_{il\,s}(n)\;\tilde{\alpha}^l_{-1}\;\alpha^i_{-1}\;|p,0\rangle \tag{3.19}$$

and the dilaton

$$|p,1\rangle \equiv P_{il}(n)\;\tilde{\alpha}^l_{-1}\;\alpha^i_{-1}\;|p,0\rangle \tag{3.20}$$

(we will refer from now on to the states $|p,0\rangle$ simply as $|p\rangle$ unless said otherwise) where $p^2 = 0,\ s = 1,2$ labels the graviton helicity, $P^{il}_s(n),\ 1 \le i,l \le 3$ projects onto the spin 2 graviton states and $P^{il}(n)$ onto the (scalar) dilaton state, and the projection operators satisfy [84]

$$\begin{aligned}
P^{il}_s(n) &= P^{li}_s(n) \quad , \quad n_i\,P^{il}_s(n) = 0 \\
P^l_{l\,s}(n) &= 0 \quad , \quad P^{il}_s(n)P_{il\,s'}(n) = 2\delta_{ss'} \\
P^{il}_s(n) &= P^{il}_s(-n) \quad , \\
P^{il}(n) &= P^{li}(n) \quad , \quad n_i\,P^{il}(n) = 0
\end{aligned}$$

$$P^{il}(n)P_{il}(n) \;\; = \;\; 2 \quad , \quad P^{il}(n) = P^{il}(-n) \quad .$$

A possible representation for these operators is as follows: first we introduce the orthonormal unit vectors:

$$\frac{n_i}{n} = (\sin\gamma\cos\phi, \sin\gamma\sin\phi, \cos\gamma),$$

$$l_i = (\sin\phi, -\cos\phi, 0),$$

$$m_i = \pm(\cos\phi\cos\gamma, \sin\phi\cos\gamma, -\sin\gamma),$$

$+$ for $\gamma < \pi/2$, $-$ for $\gamma > \pi/2$. Thus, for $P^{il}_s(n)$ we have:

$$\overset{1}{P^{il}}(n) = (l_i l_j - m_i m_j),$$

and

$$\overset{2}{P^{il}}(n) = (l_i m_j + l_j m_i).$$

For $P^{il}(n)$ we have:

$$P^{il}(n) = \delta^{il} - \frac{n^i n^l}{n^2}.$$

For the open string we have the massless vector states (photons), which are given by

$$|p,1\rangle \equiv P^{il}(n)\; \alpha^l_{-1}\, |p,0\rangle \; .$$

It is clear from eq.(3.11), since it is (as we have said before) already normal ordered, that the only oscillation modes that need to be kept in $e^{-i\lambda\cdot X_+}$ and $e^{-i\lambda\cdot X_-}$ are the ones with $n = 1$ since all the other modes commute with α_{-1} (α_1) and $\tilde{\alpha}_{-1}$ ($\tilde{\alpha}_1$) and will annihilate the ground state. Thus, we can write for gravitons

$$\langle\Psi|e^{-i\lambda\cdot X_-}$$

and

$$e^{-i\lambda\cdot X_+}|\Psi\rangle$$

as

$$\langle 1,p|e^{-i\lambda\cdot X_-} = \langle p_1|\tilde{\alpha}^l_1\alpha^i_1\; P_{il\,s}(1)e^{-i\lambda\cdot X_-} \tag{3.21}$$

and

$$e^{-i\lambda\cdot X_+}|\Psi\rangle = e^{-i\lambda\cdot X_+}P_{ils'}(2)\tilde{\alpha}^j_{-1}\alpha^m_{-1}|p_2\rangle \tag{3.22}$$

respectively. We can now compute the action of $e^{-i\lambda\cdot X_+}$ on $|\Psi\rangle$ and of $e^{-i\lambda\cdot X_-}$ on $\langle\Psi|$ with the help of the identities:

$$\left[\alpha^\mu_n, e^{-\lambda\cdot X_-}\right] = -\frac{\lambda^\mu}{\sqrt{2}}e^{i(\tau-\sigma)}\; e^{-\lambda\cdot X_-} \tag{3.23}$$

$$\left[\tilde{\alpha}_n^\mu, e^{-\lambda\cdot X_-}\right] = -\frac{\lambda^\mu}{\sqrt{2}} e^{i(\tau+\sigma)}\ e^{-\lambda\cdot X_-} \tag{3.24}$$

$$\left[e^{\lambda\cdot X_+}, \alpha_{-n}^\mu\right] = \frac{\lambda^\mu}{\sqrt{2}} e^{-i(\tau-\sigma)}\ e^{-\lambda\cdot X_+} \tag{3.25}$$

$$\left[e^{\lambda\cdot X_+}, \tilde{\alpha}_{-n}^\mu\right] = \frac{\lambda^\mu}{\sqrt{2}} e^{-i(\tau+\sigma)}\ e^{-\lambda\cdot X_+} \tag{3.26}$$

obtaining:

$$\langle\Psi|e^{-i\lambda\cdot X_-} = \langle p_1|(\tilde{\alpha}_1^l\alpha_1^i - \frac{\lambda^l}{\sqrt{2}}e^{i(\tau+\sigma)}\alpha_1^i - \frac{\lambda^i}{\sqrt{2}}e^{i(\tau-\sigma)}\tilde{\alpha}_1^l + \frac{\lambda^l\lambda^i}{2}e^{2i\tau})\ P_{il\,s}(1) \tag{3.27}$$

and

$$e^{-i\lambda\cdot X_+}|\Psi\rangle = P_{il\,s'}(2)(\tilde{\alpha}_{-1}^j\alpha_{-1}^m + \frac{\lambda^j}{\sqrt{2}}e^{-i(\tau+\sigma)}\alpha_{-1}^m + \frac{\lambda^m}{\sqrt{2}}e^{-i(\tau-\sigma)}\tilde{\alpha}_{-1}^j + \frac{\lambda^j\lambda^m}{2}e^{-2i\tau})|p_2\rangle. \tag{3.28}$$

$P_{il\,s'}(2)$ and $P_{il\,s}(1)$ refer to the projection operator needed for the initial (2) and final (1) states respectively. Substituting expressions (3.27) and (3.28) into eq.(3.11) and the equivalent expressions for $\hat{T}_b^{\mu\nu}$ and $\hat{T}_c^{\mu\nu}$ and taking the matrix elements we obtain for the graviton:

$$\begin{aligned}
\frac{\langle p_1|\tilde{\alpha}_1^l\alpha_1^i\hat{T}^{\mu\nu}(x)\tilde{\alpha}_{-1}^j\alpha_{-1}^m|p_2\rangle}{\langle\Psi|\Psi\rangle} \quad = \quad & \frac{1}{6\pi\langle\Psi|\Psi\rangle}\int\frac{d^4\lambda}{(2\pi)^4}d^4p_1d^4p_2\ d\sigma d\tau\ P_{il\,s}(1)\ P_{jm\,s'}(2)\ A^{ljim}\times \\
& [p_1^\mu p_1^\nu + p_2^\mu p_2^\nu + \frac{1}{2}\,(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu)]\langle p_1|e^{i\lambda\cdot x}e^{-i\lambda\cdot X_{cm}}|p_2\rangle\times \\
& \frac{\varphi^*(\vec{p}_1)}{E_1}\ \frac{\varphi(\vec{p}_2)}{E_2}\ \delta(p_1^0 - \sqrt{\vec{p}_1^2 + m^2})\ \delta(p_2^0 - \sqrt{\vec{p}_2^2 + m^2})\ ,
\end{aligned} \tag{3.29}$$

where

$$A^{ljim} = 4\delta^{lj}\delta^{im} - 2\delta^{lj}\lambda^i\lambda^m - 2\delta^{im}\lambda^l\lambda^j + \lambda^l\lambda^j\lambda^i\lambda^m.$$

A similar expression is found for the dilaton case. For the open string case we obtain through a similar calculation:

$$\begin{aligned}
\frac{\langle p_1|\alpha_1^i\hat{T}^{\mu\nu}(x)\alpha_{-1}^m|p_2\rangle}{\langle\Psi|\Psi\rangle} \quad = \quad & \frac{2}{3\pi\langle\Psi|\Psi\rangle}\int d^4p_1d^4p_2\ d^4\lambda\ d\tau\ \frac{e^{i\lambda\cdot x}}{(2\pi)^3}P_l^i(1)\ P_i^l(2)\times \\
& \Big[\delta^{im}\ \left(p_1^\mu p_1^\nu + p_2^\mu p_2^\nu + \frac{1}{2}\,(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu)\right) \\
& -\frac{3}{2}\eta^{\mu\nu}\lambda^i\lambda^m\Big]\ \langle p_1|e^{i\lambda\cdot x}e^{-i\lambda\cdot X_{cm}}|p_2\rangle\times \\
& \frac{\varphi^*(\vec{p}_1)}{E_1}\frac{\varphi(\vec{p}_2)}{E_2}\ \delta(p_1^0 - \sqrt{\vec{p}_1^2 + m^2})\delta(p_2^0 - \sqrt{\vec{p}_2^2 + m^2}),
\end{aligned} \tag{3.30}$$

having performed the σ integration.

Substituting eq.(3.13) into eqs.(3.29) and (3.30) and integrating over λ , p_1^0 and p_2^0, we find

$$\frac{\langle\Psi|\hat{T}^{\mu\nu}(x)_{\mathcal{A}}|\Psi\rangle}{\langle\Psi|\Psi\rangle} = \frac{T}{6(2\pi)^3\langle\Psi|\Psi\rangle}\int d^3p_1 d^3p_2 e^{i(p_2-p_1)\cdot x}\ \mathcal{B}^{\mu\nu}_{\mathcal{A}}\ \varphi^*(\vec{p}_1)\ \varphi(\vec{p}_2)\left(\frac{E_1+E_2}{(E_1E_2)^2}\right), \tag{3.31}$$

with $\mathcal{A}=(g,d,ph)$ where g,d,ph refer to the graviton, dilaton or photon case. With this notation we have that

$$\begin{aligned}\mathcal{B}^{\mu\nu}_g &= [p_1^\mu p_1^\nu + p_2^\mu p_2^\nu + \frac{1}{2}\,(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu)] \times \\ &\Bigg[4 \overset{s}{P}_{jm}(1)\ \overset{s'}{P}{}^{jm}(2) + 2\ \overset{s}{P}{}^{il}(1)\ \overset{s'}{P}{}_l^m(2) p_2^i p_1^m + 2\ \overset{s}{P}{}^{il}(1)\ \overset{s'}{P}{}_i^j(2) p_2^l p_1^j \\ &+ \overset{s}{P}{}^{il}(1)\ \overset{s'}{P}{}^{jm}(2) p_2^l p_1^j p_2^i p_1^m\Bigg],\end{aligned} \tag{3.32}$$

$$\begin{aligned}\mathcal{B}^{\mu\nu}_d &= [p_1^\mu p_1^\nu + p_2^\mu p_2^\nu + \frac{1}{2}\,(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu)] \times \\ &\Big[4P_{jm}(1)P^{jm}(2) + 2P^{il}(1)P_l^m(2)p_2^i p_1^m + 2P^{il}(1)P_i^j(2)p_2^l p_1^j \\ &+P^{il}(1)P^{jm}(2)p_2^l p_1^j p_2^i p_1^m\Big]\end{aligned} \tag{3.33}$$

and

$$\begin{aligned}\mathcal{B}^{\mu\nu}_{ph} &= \Bigg[P_l^i(1)P_i^l(2)\left(p_1^\mu p_1^\nu + p_2^\mu p_2^\nu + \frac{1}{2}\,(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu)\right) \\ &-\frac{3}{2}\eta^{\mu\nu}P_l^i(1)P^{lm}(2)(p_2^i - p_1^i)(p_2^m - p_1^m)\Bigg].\end{aligned} \tag{3.34}$$

It is convenient to extend at this point the definition of $\mathcal{B}^{\mu\nu}_{\mathcal{A}}$ to include also the tachyon case:

$$\mathcal{B}^{\mu\nu}_{ty1} = 4[p_1^\mu p_1^\nu + p_2^\mu p_2^\nu + \frac{1}{2}\,(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu)] \tag{3.35}$$

for closed strings, whreas for open strings we obtain

$$\mathcal{B}^{\mu\nu}_{ty2} = [p_1^\mu p_1^\nu + p_2^\mu p_2^\nu + \frac{1}{2}\,(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu)]. \tag{3.36}$$

There is an important restriction on the allowed combination of the momenta p_1 and p_2 that emerges when we demand the physical condition of energy-momentum conservation. This is equivalent to

$$\begin{aligned}\frac{\langle\Psi|\hat{T}^{\mu\nu}(x)_{\mathcal{A}}|\Psi\rangle_{,\nu}}{\langle\Psi|\Psi\rangle} &= \frac{iT}{6(2\pi)^3\langle\Psi|\Psi\rangle}\int d^3p_1 d^3p_2 e^{i(p_2-p_1)\cdot x}\ (p_2-p_1)_\nu \mathcal{B}^{\mu\nu}_{\mathcal{A}}\ \varphi^*(\vec{p}_1)\ \varphi(\vec{p}_2) \times \\ &\left(\frac{E_1+E_2}{(E_1E_2)^2}\right) = 0.\end{aligned} \tag{3.37}$$

Thus energy-momentum conservation tells us that only a subset of the states we are considering are physical. The constraint we have obtained in order to satisfy energy-momentum conservation is consistent with

$$(p_2 - p_1)_\nu \mathcal{B}^{\mu\nu}_{\mathcal{A}} = 0, \tag{3.38}$$

which reduces in all cases to

$$(p_2^\nu - p_1^\nu)(p_{2\,\mu} p_1^\mu - m^2) = 0. \tag{3.39}$$

This expression tell us that the 4-vectors p_1 and p_2 must satisfy either:

$$p_1^\mu = p_2^\mu \tag{3.40}$$

which is a trivial solution and which we will not be considering, or

$$p_{2\mu} p_1^\mu = m^2. \tag{3.41}$$

If p_1^μ and p_2^μ satisfy either eq.(3.40) or eq.(3.41) then energy-momentum conservation is ensured. It should be noted that the imposition of eq.(3.41), rather than eq.(3.40) is rather speculative. One point we should also notice is the fact that the imposition of eq.(3.40) would lead us to the classical energy-momentum tensor for point particles following a Gaussian distribution.

Let us consider now the $\langle \hat{T}^{\mu\nu}(x) \rangle$ for the massless string states. In order to satisfy eq.(3.41), $\vec{p}_1$ and $\vec{p}_2$ must be parallel to each other. Thus, we can now write expression eq.(3.31) in terms of eqs.(3.32)-(3.34) as:

$$\langle \hat{T}^{\mu\nu}(x) \rangle = \frac{2T}{3(2\pi)^3 \langle \Psi | \Psi \rangle} \int d^3p_1 d^3p_2 \; e^{i(p_2 - p_1)\cdot x} \; [p_1^\mu p_1^\nu + p_2^\mu p_2^\nu + \frac{1}{2}\left(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu\right)] \times \varphi^*(\vec{p}_1)\varphi(\vec{p}_2) \; \left(\frac{E_1 + E_2}{(E_1 E_2)^2}\right) \tag{3.42}$$

where, since $\vec{p}_1$ is parallel to $\vec{p}_2$, all the terms involving $P(n) \cdot \vec{p}$ have vanished. The integrals above are now restricted integrals, we are now considering only $\vec{p}_1$ and $\vec{p}_2$ which are parallel to each other.

3.2.1 Conformal Invariance

Closely related to the issue of energy-momentum conservation is that of conformal invariance. We now proceed to establish the conformal invariance of our results.

Conformal transformations are simpler to describe in the coordinates

$$x^{\pm} \equiv \frac{1}{\sqrt{2}}(\sigma \pm \tau) \; .$$

The classical energy-momentum tensor eq.(3.1) then takes the form

$$T^{\mu\nu}(x) = \frac{1}{2\pi\alpha'} \int dx^+ dx^- \; \int \frac{d^4\lambda}{(2\pi)^4} \; e^{i\lambda\cdot x} \; \partial_+ X^\mu \partial_- X^\nu \; e^{-i\lambda\cdot X(\sigma,\tau)} \; . \tag{3.43}$$

This expression is clearly invariant under the conformal transformations

$$x^+ \to f^+(x^+) \quad , \quad x^- \to f^-(x^-)$$

for arbitrary $f^{\pm}$. In other words, conformal invariance holds if the integrand $\partial_+ X^\mu \partial_- X^\nu$ has $(1,1)$ as conformal weights.

At the quantum level it is enough to require conformal invariance on-shell. That is, the matrix elements of $\hat{T}^{\mu\nu}(x)$ [eq. (3.5)] on physical states must be conformally invariant. We must then look to eq.(3.42) and show that the integrands have conformal weights $(1,1)$. Actually this expression is close to the matrix elements of the graviton vertex operator

$$\epsilon_{\mu\nu}(\lambda) \; \partial_+ X^\mu \partial_- X^\nu \; e^{i\lambda\cdot x} \; .$$

This operator has conformal weights $(1,1)$ provided $\lambda^2 = 0$ and $\lambda^\mu \epsilon_{\mu\nu}(\lambda) = 0 = \lambda^\nu \epsilon_{\mu\nu}(\lambda)$.

We see in our calculation that the fact that we are working with states with $p_1^\mu p_{\mu\, 2} = 0$ in eq. (3.42), set $\lambda = p_2 - p_1$ to be light-like : $\lambda^2 = 0$. However, we used the full operator $\partial_+ X^\mu \partial_- X^\nu$, which, generally speaking, may contain in Fourier space a longitudinal part of the form

$$\gamma_+^\mu \; \lambda^\nu + \gamma_-^\nu \; \lambda^\mu,$$

where $\gamma_\pm^\mu$ are some vectors, plus the transverse part. We can find longitudinal pieces by contracting the integrand in eq.(3.42) with λ^ν and with λ^μ separately. It is not difficult to see that the result is identically zero. Hence, the whole integrands come from the transverse part of $\partial_+ X^\mu \partial_- X^\nu$ with conformal weights $(1,1)$. This completes the proof of the conformal invariance.

It must be noticed that the same transversality property of the integrand ensures the energy-momentum conservation $\partial_\mu \hat{T}^{\mu\nu}(x) = 0$. This reflects the connection between world-sheet conformal invariance and space-time invariances in String Theory.

3.3 The classical energy-momentum tensor for point-particles

It is to be noticed that when we choose the wave-packets $\varphi(\vec{p})$ to have zero width, that is, when the state is completely localised in momentum space: $\varphi(\vec{p}) = A\delta(\vec{p}-\vec{p}_a)$, we recover the usual results found in [10] as a short calculation shows. This adds confidence to the calculations we are about to perform in the next sections. From eq.(3.42) we have:

$$\langle \hat{T}^{\mu\nu}(x)\rangle = \frac{2T|A|^2}{3(2\pi)^3\langle\Psi|\Psi\rangle}\int d^3p_1 d^3p_2\ e^{i(p_2-p_1)\cdot x}\ [p_1^\mu p_1^\nu + p_2^\mu p_2^\nu + \frac{1}{2}\,(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu)] \times \\ \delta(\vec{p}_1-\vec{p}_a)\delta(\vec{p}_2-\vec{p}_a)\ \left(\frac{E_1+E_2}{(E_1E_2)^2}\right) \quad (3.44)$$

which becomes

$$\langle \hat{T}^{\mu\nu}(x)\rangle = \frac{2TA^2}{3(2\pi)^3\langle\Psi|\Psi\rangle}\left(\frac{6p^\mu p^\nu}{E^3}\right), \quad (3.45)$$

and

$$\begin{aligned} \langle\Psi|\Psi\rangle \quad &= \quad A^2\int \frac{d^4p_1}{E_1}\frac{d^4p_2}{E_2}\ P(\vec{p}_1)\cdot P(\vec{p}_2)\ \delta(\vec{p}_1-\vec{p}_a)\ \delta(\vec{p}_2-\vec{p}_a)\ \delta(\vec{p}_1-\vec{p}_2)\times \\ &\quad \delta(p_1^0-p_2^0)\ \delta(p_1^0-\sqrt{\vec{p}_1^2+m^2})\ \delta(p_2^0-\sqrt{\vec{p}_2^2+m^2}), \end{aligned} \quad (3.46)$$

$$\langle\Psi|\Psi\rangle = A^2\int \frac{d^3p_1 d^3p_2}{E_1E_2}\ P(\vec{p}_1)\cdot P(\vec{p}_2)\ \delta(\vec{p}_1-\vec{p}_a)\ \delta(\vec{p}_2-\vec{p}_a)\ \delta(\vec{p}_1-\vec{p}_2)\ \delta(E_1-E_2), \quad (3.47)$$

$$\langle\Psi|\Psi\rangle = 2A^2\int \frac{d^3p_2}{E_aE_2}\delta(\vec{p}_2-\vec{p}_a)\ \delta(\vec{p}_2-\vec{p}_a)\ \delta(E_2-E_a), \quad (3.48)$$

$$\langle\Psi|\Psi\rangle = \frac{2A^2}{E_a^2}\delta(\vec{p}_a-\vec{p}_a)\ \delta(E_a-E_a).$$

We can regularise the Dirac deltas obtaining the following result

$$\langle\Psi|\Psi\rangle = \frac{A^2}{(2\pi)^4E_a^2}\ V\ T.$$

Putting everything together we find:

$$\langle \hat{T}^{\mu\nu}(x)\rangle = 4\pi\ \frac{p^\mu p^\nu}{V\ E}. \quad (3.49)$$

This result is nothing more than the energy-momentum tensor of a point-like particle. Looking now at the energy we have:

$$\int \frac{dV}{(2\pi)^3}\langle \hat{T}^{00}(x)\rangle \to E \quad (3.50)$$

just as it should be. A similar calculation can be performed for the tachyonic case by replacing eq.(3.42) in eq.(3.44) by either eq.(3.17) or eq.(3.18) resulting also in

$$\int \frac{dV}{(2\pi)^3}\langle \hat{T}^{00}(x)\rangle \to E.$$

3.4 $\hat{T}^{\mu\nu}(x)$ for massless string states in spherically symmetric configurations

Let us now consider the expectation value of the energy-momentum tensor for the massless closed string state given by eq.(3.42). Notice that such an expectation value is independent of the value of the particle spin (zero, one or two).

For such a case it is convenient to use the parametrisation:

$$p_1 = E_1(1, \hat{u}_1) \quad , \quad p_2 = E_2(1, \hat{u}_2)$$

with $E_1, E_2 \geq 0$ and

$$\hat{u}_1 = (\cos\phi \sin\gamma, \sin\phi \sin\gamma, \cos\gamma)$$

$$\hat{u}_2 = (\cos\beta \sin\delta, \sin\beta \sin\delta, \cos\delta).$$

But the constraint $p_1 \cdot p_2 = 0$ [eq.(3.41)] implies $\hat{u}_1 \cdot \hat{u}_2 = 1$. Thus, eq.(3.42) becomes

$$\begin{aligned} \langle \hat{T}^{\mu\nu}(x) \rangle &= \frac{2T}{3(2\pi)^3 \langle \Psi|\Psi \rangle} \int_0^\infty dE_1 \int_0^\infty dE_2 \int d\hat{u}_1 \varphi^*(E_1, \hat{u}_1)\, \varphi(E_2, \hat{u}_1)\, e^{i(E_2 - E_1)(t - \vec{x}\cdot\hat{u}_1)} \\ & \left[p_1^\mu p_1^\nu + p_2^\mu p_2^\nu + \frac{1}{2}\left(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu\right) \right] (E_1 + E_2) \end{aligned} \tag{3.51}$$

where $d\hat{u}_1 \equiv \sin\gamma\, d\gamma\, d\phi$ and we have taken into account the constraint $\hat{u}_1 \cdot \hat{u}_2 = 1$. Let us consider for simplicity spherically symmetric wave packets $\varphi(E, \hat{u}) = \varphi(E)$. If $\vec{x} = (0, 0, r)$ we can integrate over the angles with the result:

$$\begin{aligned} \langle \hat{T}^{00}(t, r) \rangle &= \frac{8T}{3(2\pi)^3 \langle \Psi|\Psi \rangle} \int_0^\infty dE_1 \int_0^\infty dE_2\, \varphi^*(E_1)\varphi(E_2)\, e^{i(E_2 - E_1)t} \\ & \frac{\sin(E_2 - E_1)r}{(E_2 - E_1)r} \left[E_1^2 + E_2^2 + E_1 E_2 \right] (E_1 + E_2) \end{aligned} \tag{3.52}$$

We can relate the result for arbitrary $x = (t, \vec{x})$ with the special case $x = (t, 0, 0, z)$ using rotational invariance as follows,

$$\begin{aligned} \langle \hat{T}^{0i}(x) \rangle &= \hat{x}^i\, C(t, r) \quad , \quad i = 1, 2, 3, \\ \langle \hat{T}^{ij}(x) \rangle &= \delta^{ij}\, A(t, r) + \hat{x}^i\, \hat{x}^j\, B(t, r) \ \ , \ i, j = 1, 2, 3. \end{aligned} \tag{3.53}$$

Here

$$C(t, r) = \langle \hat{T}^{03}(t, r = z) \rangle$$

$$A(t, r) = \langle \hat{T}^{22}(t, r = z) \rangle \quad , \quad B(t, r) = \langle \hat{T}^{33}(t, r = z) \rangle - \langle \hat{T}^{11}(t, r = z) \rangle \tag{3.54}$$

with $\hat{x}^i = \frac{x^i}{r}$ the unit vector, and

$$
\begin{aligned}
\langle \hat{T}^{11}(t, r=z) \rangle &= -\frac{8T}{3(2\pi)^3 \langle \Psi | \Psi \rangle} \int_0^\infty dE_1 \int_0^\infty dE_2 \, \varphi^*(E_1)\varphi(E_2) \, \frac{e^{i(E_2-E_1)t}}{(E_2-E_1)^2 r^2} \\
&\quad \left[\cos(E_2-E_1)r - \frac{\sin(E_2-E_1)r}{(E_2-E_1)r} \right] \left[E_1^2 + E_2^2 + E_1 E_2 \right] (E_1+E_2), \\
\langle \hat{T}^{33}(t, r=z) \rangle &= \frac{8T}{3(2\pi)^3 \langle \Psi | \Psi \rangle} \int_0^\infty dE_1 \int_0^\infty dE_2 \, \left[E_1^2 + E_2^2 + E_1 E_2 \right] (E_1+E_2) \\
&\quad \varphi^*(E_1)\varphi(E_2) \, \frac{e^{i(E_2-E_1)t}}{(E_2-E_1)r} \, \left[\sin(E_2-E_1)r + 2\frac{\cos(E_2-E_1)r}{(E_2-E_1)r} \right. \\
&\quad \left. -2\frac{\sin(E_2-E_1)r}{(E_2-E_1)^2 r^2} \right], \\
\langle \hat{T}^{03}(t, r=z) \rangle &= \frac{i8T}{3(2\pi)^3 \langle \Psi | \Psi \rangle} \int_0^\infty dE_1 \int_0^\infty dE_2 \, \left[E_1^2 + E_2^2 + E_1 E_2 \right] (E_1+E_2) \\
&\quad \varphi^*(E_1)\varphi(E_2) \, \frac{e^{i(E_2-E_1)t}}{(E_2-E_1)r} \, \left[\cos(E_2-E_1)r - \frac{\sin(E_2-E_1)r}{(E_2-E_1)r} \right] . \qquad (3.55)
\end{aligned}
$$

The other components satisfy $\langle \hat{T}^{22}(t, r=z) \rangle = \langle \hat{T}^{11}(t, r=z) \rangle$, $\langle \hat{T}^{01}(t, r=z) \rangle = \langle \hat{T}^{02}(t, r=z) \rangle = \langle \hat{T}^{12}(t, r=z) \rangle = \langle \hat{T}^{13}(t, r=z) \rangle = \langle \hat{T}^{23}(t, r=z) \rangle = 0$, as they must from rotational invariance.

As we can see the trace of the expectation value of the string energy-momentum tensor vanishes. Notice that

$$
3A(t,r) + B(t,r) = \langle \hat{T}^{00}(t,r) \rangle
$$

due to the tracelessness of the energy-momentum tensor.

The invariant functions $\langle \hat{T}^{00}(t,r) \rangle, A(t,r),\ B(t,r)$ and $C(t,r)$ in eqs.(3.52), (3.54) and (3.55) can be written in terms of outgoing and incoming waves as

$$
\begin{aligned}
\langle \hat{T}^{00}(t,r) \rangle &= \frac{1}{r} \, \left[F(t+r) - F(t-r) \right], \\
A(t,r) &= -\frac{1}{r^2} \left[H(t+r) + H(t-r) \right] + \frac{1}{r^3} \left[E(t+r) - E(t-r) \right], \\
B(t,r) &= \frac{1}{r} \, \left[F(t+r) - F(t-r) \right] + \frac{3}{r^2} \left[H(t+r) + H(t-r) \right] - \frac{3}{r^3} \left[E(t+r) - E(t-r) \right], \\
C(t,r) &= -\frac{1}{r} \left[F(t+r) + F(t-r) \right] - \frac{1}{r^2} \left[H(t+r) - H(t-r) \right] . \qquad (3.56)
\end{aligned}
$$

where

$$
\begin{aligned}
F(x) &= \frac{4T}{3(2\pi)^3 \langle \Psi | \Psi \rangle} \int_0^\infty dE_1 \int_0^\infty dE_2 \, \varphi(E_1)\varphi(E_2) \\
&\quad \frac{\sin(E_2-E_1)x}{(E_2-E_1)} \, \left[E_1^2 + E_2^2 + E_1 E_2 \right] (E_1+E_2) \, . \qquad (3.57)
\end{aligned}
$$

Notice that $F(x) = -H'(x)$ and $H(x) = E'(x)$. Let us choose now a real wave-packet $\varphi(E)$, typically peaked at $E = 0$ and decreasing rapidly away from there. For example consider a Gaussian wave-packet $\varphi(E)$:

$$\varphi(E) = N\, e^{-\alpha E^2} \tag{3.58}$$

where N is an arbitrary dimensionless constant and $\alpha \equiv 1/\sigma$ σ being the width of the wave-packet. An expression for α can be obtained by demanding the wave-functions to be correctly normalised. In particular when we consider

$$\langle\Psi|\Psi\rangle = 2T \int_0^\infty dp \frac{p^2}{E^2} |\varphi(\vec{p})|^2 = 1, \tag{3.59}$$

$$\frac{1}{2T} = \int_0^\infty dE\ |\varphi(E)|^2 = N^2 \int_0^\infty dE\ e^{-2\alpha E^2} \tag{3.60}$$

from this expression we arrive at

$$\alpha = \frac{\pi\ N^4\ T^2}{2}. \tag{3.61}$$

We see from eq.(3.56) that the energy density $\langle \hat{T}^{00}(r,t)\rangle$ and the energy flux $\langle \hat{T}^{0i}(r,t)\rangle$ behave like spherical waves describing the way a massless string state spreads out starting from the initial wave-packet we choose.

$F(x)$ is difficult to calculate exactly, but its asymptotic behaviour can be obtained. Changing the integration variables in eq.(3.57) to

$$E_2 - E_1 = v\tau/x \quad , \quad E_2 + E_1 = v\ ,$$

we find

$$\begin{aligned} F(x) \quad = \quad & \frac{T}{3(2\pi)^3\langle\Psi|\Psi\rangle} \int_0^\infty v^3\, dv \int_0^x \frac{d\tau}{\tau}\ \varphi(\frac{v}{2}[1+\frac{\tau}{x}])\varphi(\frac{v}{2}[1-\frac{\tau}{x}]) \\ & \sin(v\tau) \left[3 + \frac{\tau^2}{x^2}\right] \ . \end{aligned} \tag{3.62}$$

Now, we can let $x \to \infty$ obtaining the following integrals:

$$\int_0^\infty \frac{d\tau}{\tau}\ \sin v\tau\ e^{-\frac{\alpha v^2 \tau^2}{2x^2}} = \frac{\pi}{2}\Phi(\frac{x}{\sqrt{2\alpha}}) \tag{3.63}$$

and

$$\int_0^\infty \frac{d\tau}{x^2}\ \tau\ \sin v\tau\ e^{-\frac{\alpha v^2 \tau^2}{2x^2}} = \frac{x}{2\alpha} \sqrt{\frac{2\pi}{\alpha}}\ e^{-\frac{x^2}{2\alpha}}. \tag{3.64}$$

Substituting eqs.(3.63) and (3.64) into eq.(3.62) and taking the limit $x \to \infty$ we arrive to the result

$$F(x) \overset{x\to\pm\infty}{=} \pm \frac{1}{(2\pi)^3\alpha^2}\sqrt{\frac{2\alpha}{\pi}} + O(e^{-x^2}). \tag{3.65}$$

We find through similar calculations:

$$H(x) \stackrel{x\to\pm\infty}{=} -\frac{1}{(2\pi)^3\alpha^2}\sqrt{\frac{2\alpha}{\pi}}\ |x| + O(e^{-x^2}), \tag{3.66}$$

$$E(x) \stackrel{x\to\pm\infty}{=} -\frac{1}{2(2\pi)^3\alpha^2}\sqrt{\frac{2\alpha}{\pi}}\ x^2\ sign(x) + O(e^{-x^2}), \tag{3.67}$$

To gain an insight into the behaviour of $\langle \hat{T}^{\mu\nu}\rangle$, we consider the limiting cases: $r \to \infty,\ t$ fixed and $t \to \infty,\ r$ fixed with the following results:

a) $r \to \infty,\ t$ fixed

$$\langle \hat{T}^{00}(r,t)\rangle \stackrel{r\to\infty}{=} \frac{2}{(2\pi)^3\alpha^2}\sqrt{\frac{2\alpha}{\pi}}\frac{1}{r}, \tag{3.68}$$

$$A(r,t) \stackrel{r\to\infty}{=} \frac{1}{(2\pi)^3\alpha^2}\sqrt{\frac{2\alpha}{\pi}}\frac{1}{r}, \tag{3.69}$$

$$B(r,t) \stackrel{r\to\infty}{=} -\frac{1}{(2\pi)^3\alpha^2}\sqrt{\frac{2\alpha}{\pi}}\frac{1}{r}, \tag{3.70}$$

and

$$C(r,t) \stackrel{r\to\infty}{=} \frac{2}{(2\pi)^3\alpha^2}\sqrt{\frac{2\alpha}{\pi}}\frac{t}{r^2},. \tag{3.71}$$

b) $t \to \infty, r$ fixed

$$\langle \hat{T}^{00}(r,t)\rangle \stackrel{t\to\infty}{=} -\frac{2}{3\alpha^2(2\pi)^3}\ e^{-\frac{t^2}{2\alpha}}\ \sinh\frac{tr}{\alpha}\ \frac{t}{r}, \tag{3.72}$$

$$A(r,t) \stackrel{t\to\infty}{=} \frac{2}{\alpha(2\pi)^3}\ e^{-\frac{t^2}{2\alpha}}\ \sinh\frac{tr}{\alpha}\ \frac{t}{r^2}, \tag{3.73}$$

$$B(r,t) \stackrel{t\to\infty}{=} -\frac{2}{3\alpha^2(2\pi)^3}\ e^{-\frac{t^2}{2\alpha}}\ \sinh\frac{tr}{\alpha}\ \frac{t}{r} \tag{3.74}$$

and

$$C(r,t) \stackrel{t\to\infty}{=} -\frac{2}{3\alpha^2(2\pi)^3}\ e^{-\frac{t^2}{2\alpha}}\ \sinh\frac{tr}{\alpha}\ \frac{t}{r}. \tag{3.75}$$

Thus, as we mentioned earlier the energy density, the energy flux and the components of the stress tensor propagate as spherical outgoing waves. For t fixed, the energy density decays as r^{-1} while the energy flux decays as r^{-2}.

For r fixed and large t, the energy-momentum components decay exponentially fast as $O(e^{-t^2})$. In this regime we find a negative energy density. The spherical wave seems to leave behind a small but negative energy density. Notice that T^{00} is not a positive definite quantity for strings [see eq.(3.1)]. We will discuss this result in the next chapter.

The results in this section have been derived using a Gaussian shape for the wave packet. It is clear that the results will be qualitatively similar for any fast-decaying wave function $\varphi(E)$.

It should be noted that analogous but not identical results follow for massless scalar waves in the point particle case. A spherically symmetric solution of the wave equation in $D = 3 + 1$ dimensions

$$\partial^2 \phi(r,t) = 0$$

takes the form,

$$\phi(r,t) = \frac{1}{r}\left[f(t-r) + g(t+r)\right] \tag{3.76}$$

where $f(x)$ and $g(x)$ are arbitrary functions. The energy-momentum tensor for such a massless scalar field can be written as:

$$\hat{T}^{\mu\nu}(x)_\phi = \partial^\mu\phi\partial^\nu\phi - \frac{\eta^{\mu\nu}}{2}(\partial\phi)^2. \tag{3.77}$$

Inserting eq.(3.76) into eq.(3.77) yields,

$$\begin{aligned}
\hat{T}^{00}(r,t)_\phi &= \frac{1}{r^2}\left\{f'^2 + g'^2 + \frac{f+g}{2r}\left[2(f'-g') + \frac{f+g}{r}\right]\right\}, \\
\hat{T}^{0i}(r,t)_\phi &= \frac{x^i}{r^3}\left[g'^2 - f'^2 - \frac{1}{r}(f+g)(f'-g')\right], \\
\hat{T}^{ij}(r,t)_\phi &= \frac{1}{2r^2}\left[\delta^{ij} + \frac{2x^i x^j}{r^2}\right]\left[f' - g' + \frac{1}{r}(f+g)\right]^2.
\end{aligned}$$

We see that the string $\hat{T}^{\mu\nu}$ scales as $1/r$ [eq.(3.56)] whereas the field $\hat{T}^{\mu\nu}_\phi$ scales as $1/r^2$. Perhaps the slower $\hat{T}^{\mu\nu}$ decay for strings can be related to the fact that they are extended objects.

3.4.1 The total energy for a quantum bosonic string

The asymptotic behaviour of $\hat{T}^{00}$ which scales as $1/r$ can make us suspect that the total energy may diverge, since the total energy is given by the integral over the volume enclosing $\hat{T}^{00}$ and such an integral involves a power of r^2 in the numerator. Let us then estimate the total energy for the quantum string.

There are basically two regions where we would like to compute the total energy: the region where $r < T$ and the region where $r > T$ (remembering that T is the size of a finite temporal box and that $\alpha \sim T^2$).

a)$r > T$, T large but fixed. From eq.(3.68) we have:

$$E_T \sim \int^R \frac{dV}{\alpha^{3/2}\, r}, \tag{3.78}$$

$$E_T \sim \int^R \frac{dV}{r\, T^3}, \tag{3.79}$$

$$E_T \sim \left(\frac{R}{T}\right)^2 \frac{1}{T}. \tag{3.80}$$

b)$r < T$, T large but fixed. From eq.(3.52) we have:

$$\begin{aligned} E_T &\sim \int_0^R r^2 \langle \hat{T}^{00}(r,t)\rangle\, dr \\ &\sim T \int_0^\infty dE_1 \int_0^\infty \frac{dE_2}{(E_1 - E_2)^3} \left[\sin(E_1 - E_2)R - (E_1 - E_2)R\cos(E_1 - E_2)R\right] \times \\ &\quad \left(E_1^2 + E_1^2 + E_1 E_2\right)(E_1 + E_2)\varphi(E_1)\varphi(E_2). \end{aligned} \tag{3.81}$$

Since $\varphi(E) \sim e^{-T^2E^2}$ we see that these integrals are dominated by $E \ll 1/T$ and since $R < T$ we have

$$E_T \sim T \int_0^{\ll 1/T} dE_1 \int_0^{\ll 1/T} dE_2\; R^3 \;\left(E_1^2 + E_1^2 + E_1 E_2\right)(E_1 + E_2)\varphi(E_1)\varphi(E_2). \tag{3.82}$$

Solving this last set of integrals we obtain the following result:

$$E_T \sim \frac{R^3}{T^4} \sim \left(\frac{R}{T}\right)^3 \frac{1}{T}. \tag{3.83}$$

What we can see from these expressions is that the total energy diverges only when we work in a temporal box which is very small compared to the radius of the volume of space, but this is an acausal situation, which is not physically realisable.

In regions where $R \sim T$ (that is in the border of our causal horizon) the total energy is finite and small. When our temporal box is larger than the radius of the space volume the total energy converges to a finite small value. In the expressions above R acts like a cut-off for the size of the string source. There is another reason why we can expect finite energies to arise. The calculation is for a single string, and in reality we expect other strings to be present as well and these will provide a natural cut-off scale, as in the case of global cosmic strings.

3.5 $\hat{T}^{\mu\nu}$ for massless string states in cylindrically symmetric configurations.

Cosmic strings can be considered as essentially very long straight strings in (almost) cylindrically symmetric configurations. Although they behave as fundamental strings only

classically and in the Nambu approximation (that is, zero string thickness), it is important to study the expectation value of $\hat{T}^{\mu\nu}(x)$ for a cylindrically symmetric configuration. It would be interesting to see if there is an equivalent quantum version of the deficit angle found for cosmic strings [87]. Consider a cylindrically symmetric wave packet

$$\varphi(E, \hat{u}) = \varphi(E, \gamma). \tag{3.84}$$

We can re-write the integral eq.(3.51) in a more convenient way to analyse $\langle \hat{T}^{\mu\nu}(x) \rangle$ in a cylindrical configuration as follows:

$$\begin{aligned} \langle \hat{T}^{\mu\nu}(x) \rangle \;\; = \;\; & \frac{4T}{3(2\pi)^4 \langle \Psi | \Psi \rangle} \int_0^\infty dE_1 \int_0^\infty dE_2 \int_0^\pi d\gamma \sin\gamma \int_0^{2\pi} d\phi \; \varphi^*(E_1, \gamma) \; \varphi(E_2, \gamma) \\ & e^{i(E_2 - E_1)t} \; e^{i(E_2 - E_1)\rho \cos\phi \sin\gamma} \left[p_1^\mu p_1^\nu + p_2^\mu p_2^\nu + \frac{1}{2} \left(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu \right) \right] (E_1 + E_2) \end{aligned} \tag{3.85}$$

where we have chosen $\vec{x} = (\rho, 0, 0)$, $\hat{u}_1.\vec{x} = \rho \cos\phi \, \sin\gamma$. Integrating over ϕ in eq.(3.85) we find

$$\begin{aligned} \langle \hat{T}^{00}(t, \rho) \rangle \;\; = \;\; & \frac{4T}{3(2\pi)^3 \langle \Psi | \Psi \rangle} \int_0^\infty dE_1 \int_0^\infty dE_2 \int_0^\pi d\gamma \sin\gamma \; \varphi^*(E_1, \gamma) \varphi(E_2, \gamma) \\ & \left[E_1^2 + E_2^2 + E_1 E_2 \right] (E_1 + E_2) J_0([E_1 - E_2]\rho \, \sin\gamma) \; e^{i(E_2 - E_1)t} \; , \\ \langle \hat{T}^{33}(t, \rho) \rangle \;\; = \;\; & \frac{4T}{3(2\pi)^3 \langle \Psi | \Psi \rangle} \int_0^\infty dE_1 \int_0^\infty dE_2 \int_0^\pi d\gamma \sin\gamma \cos^2\gamma \; \varphi^*(E_1, \gamma) \varphi(E_2, \gamma) \\ & \left[E_1^2 + E_2^2 + E_1 E_2 \right] (E_1 + E_2) J_0([E_1 - E_2]\rho \, \sin\gamma) \; e^{i(E_2 - E_1)t} \; , \\ \langle \hat{T}^{03}(t, \rho) \rangle \;\; = \;\; & \frac{2T}{3(2\pi)^3 \langle \Psi | \Psi \rangle} \int_0^\infty dE_1 \int_0^\infty dE_2 \int_0^\pi d\gamma \sin\gamma \cos\gamma \; \varphi^*(E_1, \gamma) \varphi(E_2, \gamma) \\ & \left[E_1^2 + E_2^2 + E_1 E_2 \right] (E_1 + E_2) \; J_0([E_1 - E_2]\rho \, \sin\gamma) \; e^{i(E_2 - E_1)t} \; . \end{aligned} \tag{3.86}$$

Using rotational invariance around the z-axis we can express $\hat{T}^{\alpha\beta}(x)$, $\hat{T}^{3\alpha}(x)$ and $\hat{T}^{0\alpha}(x)$, $\alpha, \beta = 1, 2$ as follows

$$\begin{aligned} \hat{T}^{\alpha\beta}(x) \;\; &= \;\; \delta^{\alpha\beta} \; a(t, \rho) + e^\alpha \, e^\beta \; b(t, \rho) \\ \hat{T}^{0\alpha}(x) \;\; &= \;\; e^\alpha \; c(t, \rho) \quad , \quad \hat{T}^{3\alpha}(x) = e^\alpha \; d(t, \rho) \end{aligned}$$

where $e^\alpha = (\cos\phi, \sin\phi)$. The coefficients $a(t, \rho), b(t, \rho), c(t, \rho)$ and $d(t, \rho)$ follow from the calculation for $\rho = x$ (that is $\phi = 0$):

$$a(t, \rho) \;\; = \;\; \langle \hat{T}^{22}(t, \rho = x) \rangle \quad , \quad b(t, \rho) = \langle \hat{T}^{11}(t, \rho = x) \rangle - \langle \hat{T}^{22}(t, \rho = x) \rangle \; ,$$

$$c(t,\rho) \;=\; \langle \hat{T}^{01}(t,\rho=x)\rangle \quad , \quad d(t,\rho)=\langle \hat{T}^{13}(t,\rho=x)\rangle$$

We find from eqs.(3.84, 3.85) at $\rho = x$,

$$\begin{aligned}
\langle \hat{T}^{11}(t,\rho=x)\rangle &= \frac{2T}{3(2\pi)^3\langle\Psi|\Psi\rangle}\int_0^\infty dE_1\int_0^\infty dE_2\int_0^\pi d\gamma\sin^3\gamma\;\varphi^*(E_1,\gamma)\varphi(E_2,\gamma)\;e^{i(E_2-E_1)t}\\
&\quad \left[E_1^2+E_2^2+E_1E_2\right](E_1+E_2)\;\left[J_0([E_1-E_2]\rho\,\sin\gamma)-J_2([E_1-E_2]\rho\,\sin\gamma)\right],\\[2ex]
\langle \hat{T}^{22}(t,\rho=x)\rangle &= \frac{4T}{3(2\pi)^3\langle\Psi|\Psi\rangle}\int_0^\infty dE_1\int_0^\infty dE_2\int_0^\pi d\gamma\sin^2\gamma\;\varphi^*(E_1,\gamma)\varphi(E_2,\gamma)\\
&\quad \left[E_1^2+E_2^2+E_1E_2\right](E_1+E_2)\;J_1([E_1-E_2]\rho\,\sin\gamma)\;\frac{e^{i(E_2-E_1)t}}{(E_1-E_2)\rho}\;,\\[2ex]
\langle \hat{T}^{01}(t,\rho=x)\rangle &= \frac{i4T}{3(2\pi)^3\langle\Psi|\Psi\rangle}\int_0^\infty dE_1\int_0^\infty dE_2\int_0^\pi d\gamma\sin^2\gamma\;\varphi^*(E_1,\gamma)\varphi(E_2,\gamma)\\
&\quad \left[E_1^2+E_2^2+E_1E_2\right](E_1+E_2)\;J_1([E_1-E_2]\rho\,\sin\gamma)\;e^{i(E_2-E_1)t}\;,\\[2ex]
\langle \hat{T}^{13}(t,\rho=x)\rangle &= \frac{i4T}{3(2\pi)^3\langle\Psi|\Psi\rangle}\int_0^\infty dE_1\int_0^\infty dE_2\int_0^\pi d\gamma\sin^2\gamma\cos\gamma\;\varphi^*(E_1,\gamma)\varphi(E_2,\gamma)\\
&\quad \left[E_1^2+E_2^2+E_1E_2\right](E_1+E_2)\;J_1([E_1-E_2]\rho\,\sin\gamma)\;e^{i(E_2-E_1)t}\;,
\end{aligned}\tag{3.87}$$

where $J_n(z)$ is a Bessel function of integer order. Cylindrical symmetry also results in

$$\langle \hat{T}^{02}(t,\rho=x)\rangle = \langle \hat{T}^{12}(t,\rho=x)\rangle = \langle \hat{T}^{23}(t,\rho=x)\rangle = 0.$$

For a wave packet symmetric with respect to the xy plane,

$$\varphi(E,\gamma)=\varphi(E,\pi-\gamma)\;,$$

and we find

$$d(t,\rho)=\langle \hat{T}^{13}(t,\rho=x)\rangle = 0\;,\;\langle \hat{T}^{03}(t,\rho)\rangle = 0\;.$$

Notice that

$$2a(t,\rho)+b(t,\rho)+\langle \hat{T}^{33}(t,\rho)\rangle = \langle \hat{T}^{00}(t,\rho)\rangle$$

due to the tracelessness of the energy-momentum tensor. In order to compute the asymptotic behaviour $\rho\to\infty$, we change the integration variables in eq.(3.86)-(3.87) to

$$E_2-E_1 = v\tau/\rho \quad , \quad E_2+E_1 = v\;.$$

We find for the energy-density for a real $\varphi(E,\gamma)$,

$$\begin{aligned}
\langle \hat{T}^{00}(t,\rho)\rangle &= \frac{T}{6(2\pi)^3\langle\Psi|\Psi\rangle\,\rho}\int_0^\infty v^4\,dv\int_0^\rho d\tau\int_0^\pi \sin\gamma\;d\gamma\;\varphi(\frac{v}{2}[1+\frac{\tau}{\rho}],\gamma)\varphi(\frac{v}{2}[1-\frac{\tau}{\rho}],\gamma)\\
&\quad \cos(v\tau t/\rho)\;\left[3-\frac{\tau^2}{\rho^2}\right]\;J_0(v\tau\sin\gamma)\;.
\end{aligned}$$

This representation is appropriate to compute the limit $\rho \to \infty$ with t fixed. We find in such a limit,

$$\langle \hat{T}^{00}(t,\rho)\rangle \overset{\rho\to\infty}{=} \frac{T}{(2\pi)^3\langle\Psi|\Psi\rangle\ \rho} \int_0^\infty E^3 dE \int_{\sin\gamma > t/\rho} d\gamma\ \frac{\varphi(E,\gamma)^2}{\sqrt{1-\left(\frac{t}{\rho\sin\gamma}\right)^2}}\ ,$$

where we have used the formula

$$\int_0^\infty J_0(ax)\ \cos bx\ dx = \frac{\theta(a^2-b^2)}{\sqrt{a^2-b^2}}\ .$$

That is, the energy decays as ρ^{-1} for large ρ and fixed t.

In an analogous manner we compute the $\rho \to \infty$, t fixed behaviour of the other components of $\langle \hat{T}^{\mu\nu}(x)\rangle$ with the following results:

$$\begin{aligned}
a(t,\rho) &\overset{\rho\to\infty}{=} \frac{8T}{(2\pi)^3\langle\Psi|\Psi\rangle\ \rho} \int_0^\infty E^3 dE \int_{\sin\gamma > t/\rho} d\gamma\ \varphi(E,\gamma)^2 \\
&\qquad \sin^2\gamma\ \sqrt{1-\left(\frac{t}{\rho\sin\gamma}\right)^2}\ , \\
a(t,\rho)+b(t,\rho) &\overset{\rho\to\infty}{=} \frac{8T\ t^2}{(2\pi)^3\langle\Psi|\Psi\rangle\ \rho^3} \int_0^\infty E^3 dE \int_{\sin\gamma > t/\rho} d\gamma\ \frac{\varphi(E,\gamma)^2}{\sqrt{1-\left(\frac{t}{\rho\sin\gamma}\right)^2}} \\
&\overset{\rho\to\infty}{=} \frac{t^2}{\rho^2}\ \langle \hat{T}^{00}(t,\rho)\rangle\ , \\
\langle \hat{T}^{33}(t,\rho)\rangle &\overset{\rho\to\infty}{=} \frac{8T}{(2\pi)^3\langle\Psi|\Psi\rangle\ \rho} \int_0^\infty E^3 dE \int_{\sin\gamma > t/\rho} d\gamma \cos^2\gamma\ \frac{\varphi(E,\gamma)^2}{\sqrt{1-\left(\frac{t}{\rho\sin\gamma}\right)^2}}\ , \\
c(t,\rho) &\overset{\rho\to\infty}{=} \frac{8T\ t}{(2\pi)^3\langle\Psi|\Psi\rangle\ \rho^2} \int_0^\infty E^3 dE \int_{\sin\gamma > t/\rho} d\gamma\ \frac{\varphi(E,\gamma)^2}{\sqrt{1-\left(\frac{t}{\rho\sin\gamma}\right)^2}} \\
&\overset{\rho\to\infty}{=} \frac{t}{\rho}\ \langle \hat{T}^{00}(t,\rho)\rangle\ .
\end{aligned}$$

The calculation of the $t \to \infty$ behaviour for ρ fixed can be easily obtained from eq.(3.85). That is,

$$\begin{aligned}
\langle \hat{T}^{00}(t,\rho)\rangle = \frac{4T}{3(2\pi)^4\langle\Psi|\Psi\rangle} \int_0^\pi d\gamma \sin\gamma \int_0^{2\pi} d\phi\, [f_3(\gamma,\phi)f_0^*(\gamma,\phi) + 2f_1(\gamma,\phi)f_2^*(\gamma,\phi) \\
+2f_2(\gamma,\phi)f_1^*(\gamma,\phi)f_0(\gamma,\phi)f_3^*(\gamma,\phi)]
\end{aligned} \tag{3.88}$$

where

$$f_n(\gamma,\phi) \equiv \int_0^\infty E^n\ dE\ \varphi(E,\gamma)\ e^{iE(\rho\sin\gamma\sin\phi - t)}\ . \tag{3.89}$$

We explicitly see here $\langle \hat{T}^{00}(t,\rho)\rangle$ as a superposition of outgoing and ingoing cylindrical waves. As before, a typical form of the wave function $\varphi(E,\gamma)$ is a Gaussian Ansatz

$$\varphi(E,\gamma) = N\ e^{-E^2(\alpha^2\sin^2\gamma+\beta^2\cos^2\gamma)} \tag{3.90}$$

where N is an arbitrary dimensionless constant. An expression for α and β can be obtained by demanding the wave-functions to be correctly normalised. In particular when we consider $\langle\Psi|\Psi\rangle = 1$, we obtain (analogously to eq.(3.59))

$$\langle\Psi|\Psi\rangle = T\int_0^\infty dp\ \sin\gamma\ d\gamma\ \frac{p^2}{E^2}|\varphi(\vec{p})|^2 = 1, \tag{3.91}$$

hence from eq.(3.90),

$$\langle\Psi|\Psi\rangle = TN^2\int_0^\infty dE\ \sin\gamma\ d\gamma\ e^{-2E^2(\alpha^2\sin^2\gamma+\beta^2\cos^2\gamma)}. \tag{3.92}$$

Making the following change of variable:

$$u = \cos\gamma,$$

we obtain

$$\frac{1}{N^2\ T} = \int_0^\infty dE\int_{-1}^{1} du\ e^{-2E^2[\alpha^2+u^2(\beta^2-\alpha^2)]},$$
$$\frac{1}{N^2\ T} = \frac{1}{2A}\sqrt{\frac{\pi}{2}}\int_{-1}^{1}\frac{du}{\sqrt{u^2+\alpha^2/A^2}},$$

where $A=(\beta^2-\alpha^2)^{1/2}$, therefore $\beta>\alpha$. Thus, we find

$$\frac{1}{N^2\ T} = \sqrt{\frac{\pi}{2(\beta^2-\alpha^2)}}\sinh^{-1}\left(\frac{\sqrt{\beta^2-\alpha^2}}{\alpha}\right),$$
$$\sinh\left(\frac{\sqrt{2/\pi(\beta^2-\alpha^2)}}{N^2\ T}\right) = \frac{\sqrt{\beta^2-\alpha^2}}{\alpha}.$$

If $\beta^2-\alpha^2 \ll T^2$ we obtain

$$\alpha \simeq \sqrt{\frac{\pi}{2}}N^2\ T$$

whilst β must satisfy $\beta^2>\alpha^2$ and $(\beta^2-\alpha^2)\ll T^2$.

Returning to the integral (3.89), we can see that this integral is dominated in the $t\to\infty$ limit by its lower bound. We find,

$$f_n(\gamma,\phi) \overset{t\to\infty}{=} n!(-i)^{n+1}\ \varphi(0,\gamma)\ t^{-n-1}\left[1+O(t^{-1})\right]+O(e^{-t^2})\ .$$

Inserting this result in eq.(3.88) yields

$$\langle\hat{T}^{00}(t,\rho)\rangle \overset{t\to\infty}{=} 0 - O(e^{-t^2})\ .$$

Using the same technique we find the $t \to \infty$ behaviour ρ fixed, for the other $\langle \hat{T}^{\mu\nu}(t,\rho) \rangle$ components,

$$\begin{aligned}
\langle \hat{T}^{33}(t,\rho) \rangle &\overset{t\to\infty}{=} 0 - O(e^{-t^2}), \\
a(t,\rho) &\overset{t\to\infty}{=} 0 - O(e^{-t^2}), \\
b(t,\rho) &\overset{t\to\infty}{=} 0 + O(e^{-t^2}), \\
c(t,\rho) &\overset{t\to\infty}{=} 0 - O(e^{-t^2}).
\end{aligned}$$

From the expressions above we can see that for large t and ρ fixed the energy density decays exponentially fast as $O(e^{-t^2})$ leaving behind a rapidly vanishing negative contribution just as in the spherical case. For ρ large and fixed t the situation is similar to the spherical one as well, the energy density decays as $1/\rho$ and the energy flux decays as $1/\rho^2$.

3.6 Final remarks.

An important point we need to stress, is the fact that, in contrast to what happens in the classical theory where there is a logarithmic divergence as we approach the core of the string [85], in the quantum theory we can obtain a metric where, if any divergences are present these are not related at all to the position of the string. We showed, that the localised behaviour of the string given by $\delta(x - X(\sigma,\tau))$ disappears when we take into account the quantum nature of the strings. In the *classical theory of radiating strings*, divergences coincide with the source of the gravitational field. However, as we have seen, quantum mechanically this is not the case. The source position is smeared by the quantum fluctuations. The probability amplitude for the centre of mass string position is given by the Fourier transform of $\varphi(\vec{p})$ and is a smooth function peaked at the origin.

CHAPTER 4

Another derivation for $\langle \hat{T}^{\mu\nu}(r,t) \rangle$

In the previous chapter we computed the expectation value for the energy-momentum tensor both in spherically and cylindrically symmetric configurations. We presented exact asymptotic results for $\langle \hat{T}^{\mu\nu}(r,t) \rangle$ in several interesting regions of space-time. We are interested in calculating also the gravitational field arising from the quantum string, so the need for an explicit expression for the $\langle \hat{T}^{\mu\nu}(r,t) \rangle$ components is apparent. In this chapter, we will try to obtain this by computing $\langle \hat{T}^{\mu\nu}(r,t) \rangle$, for the spherically symmetric case, in an approximate way which will allow us to work with this expectation value for all r and t.

4.1 The $\alpha-$large approximation

Our approximation will be based on the following observations: first, in the spherical case we know that the functions $F(z)$, $H(z)$ and $E(z)$ defined in the last chapter do not depend on the angles ϕ and γ which defined our spherical configuration; second, the wave-packets $\varphi(\vec{p})$ have a Gaussian ansatz; in other words, they are fast decaying functions, the width of the packet being given by $1/\alpha$. (This α is not to be confused with α' in the string tension.) If we consider that most of this distribution has a width which is very small in momentum space i.e. an almost monochromatic wave-function (monochromatic in the sense that because we are working in a massless case, any distribution in momentum is basically a distribution in energy), then $1/\alpha$ has to be very small (α is large). We believe that working in this approximation makes physical sense.

With this approximation it is convenient to make the following change of variables

$$u = E_2 - E_1 \quad , v = E_2 + E_1$$

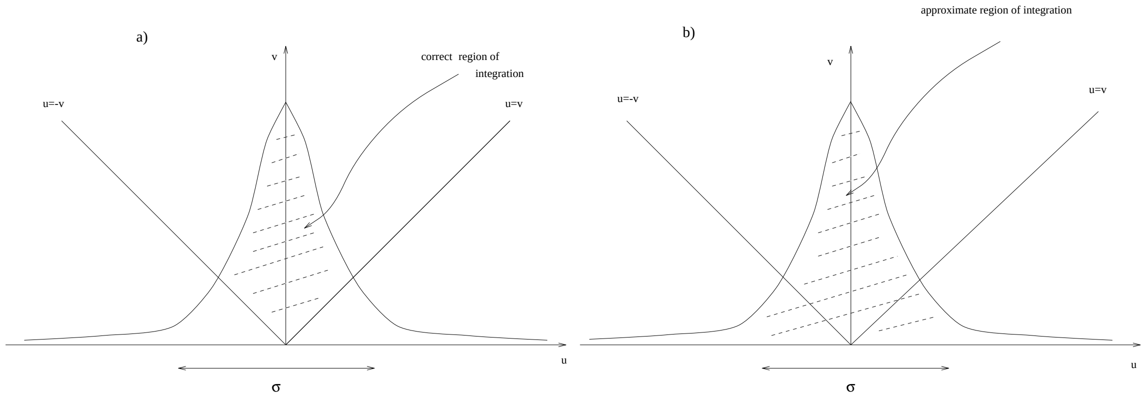

Figure 4.1: a) The integration region for the integral $F(z)$. It is exceedingly difficult to integrate explicitly over such a region. In contrast we have in b) that if the width of the wave function σ is sufficiently small we can integrate over the whole range: $-\infty \leq u \leq \infty$ explicitly introducing small errors. To have a wave-packet $\varphi(E)$ with small σ means that most particles in the particle ensemble posses the same energy.

Eq.(3.57) then becomes

$$F(z) = \frac{1}{6(2\pi)^3} \sqrt{\frac{2\alpha}{\pi}} \int_0^{\infty} dv \int_{-v}^{v} du \; e^{-\frac{\alpha}{2}(u^2+v^2)} \frac{\sin uz}{u} \left[3v^3 + vu^2\right] \tag{4.1}$$

Now, in the approximation of α being large, we can change the limits of integration over u (see fig.(4.1)). Thus eq.(4.1) can be written as

$$F(z) = \frac{1}{3(2\pi)^3} \sqrt{\frac{2\alpha}{\pi}} \int_0^{\infty} dv \int_{0}^{\infty} du \; e^{-\frac{\alpha}{2}(u^2+v^2)} \frac{\sin uz}{u} \left[3v^3 + vu^2\right]. \tag{4.2}$$

We can now solve both integrals (a list of important integrals are presented in appendix B). After some work we obtain the following result

$$F(z) = \frac{1}{\alpha^2(2\pi)^3} \sqrt{\frac{2\alpha}{\pi}} \left[\pi \Phi(\frac{z}{\sqrt{2\alpha}}) + \frac{z}{6} \sqrt{\frac{2\pi}{\alpha}} \; e^{-\frac{z^2}{2\alpha}}\right]. \tag{4.3}$$

Where $\Phi(x)$ is the probability integral as defined in appendix B. Integrating eq.(4.3) over z and multiplying by -1, we obtain:

$$H(z) = -\frac{\pi}{\alpha^2(2\pi)^3} \sqrt{\frac{2\alpha}{\pi}} \left[z\Phi(\frac{z}{\sqrt{2\alpha}}) + \frac{5}{6} \sqrt{\frac{2\alpha}{\pi}} \; e^{-\frac{z^2}{2\alpha}}\right] \tag{4.4}$$

and integrating this expression over z, we obtain

$$E(z) = -\frac{\pi}{2\alpha^2(2\pi)^3} \sqrt{\frac{2\alpha}{\pi}} \left[\Phi(\frac{z}{\sqrt{2\alpha}}) \left(z^2 + \frac{2\alpha}{3}\right) + z\sqrt{\frac{2\alpha}{\pi}} \; e^{-\frac{z^2}{2\alpha}}\right]. \tag{4.5}$$

In eqs.(4.4)-(4.5) we have used the fact that we know the exact asymptotic behaviour of these functions in order to set the arbitrary integration constant to zero.

4.2 Behaviour of $\langle \hat{T}^{\mu\nu}(r,t)\rangle$ when r and $t \to \infty$

Now let us compute once more the limit of $\langle \hat{T}^{\mu\nu}(r,t)\rangle$ when r and $t \to \infty$. The results we obtain in these limits are:

a) t fixed, $r \to \infty$

$$\langle \hat{T}^{00}(r,t)\rangle \overset{r\to\infty}{=} \frac{2}{\alpha^2(2\pi)^3}\sqrt{\frac{2\alpha}{\pi}}\,\frac{1}{r},$$

$$A(r,t) \overset{r\to\infty}{=} \frac{1}{\alpha^2(2\pi)^3}\sqrt{\frac{2\alpha}{\pi}}\,\frac{1}{r},$$

$$B(r,t) \overset{r\to\infty}{=} -\frac{1}{\alpha^2(2\pi)^3}\sqrt{\frac{2\alpha}{\pi}}\,\frac{1}{r}$$

and

$$C(r,t) \overset{r\to\infty}{=} \frac{2}{\alpha^2(2\pi)^3}\sqrt{\frac{2\alpha}{\pi}}\,\frac{t}{r^2}.$$

b) r fixed, $t \to \infty$

$$\langle \hat{T}^{00}(r,t)\rangle \overset{t\to\infty}{=} -\frac{2}{3\alpha^2(2\pi)^3}\,e^{-\frac{t^2}{2\alpha}}\,\sinh\frac{tr}{\alpha}\,\frac{t}{r},$$

$$A(r,t) \overset{t\to\infty}{=} \frac{2}{\alpha(2\pi)^3}\,e^{-\frac{t^2}{2\alpha}}\,\sinh\frac{tr}{\alpha}\,\frac{t}{r^2},$$

$$B(r,t) \overset{t\to\infty}{=} -\frac{2}{3\alpha^2(2\pi)^3}\,e^{-\frac{t^2}{2\alpha}}\,\sinh\frac{tr}{\alpha}\,\frac{t}{r}$$

and

$$C(r,t) \overset{t\to\infty}{=} -\frac{2}{3\alpha^2(2\pi)^3}\,e^{-\frac{t^2}{2\alpha}}\,\sinh\frac{tr}{\alpha}\,\frac{t}{r}.$$

As we can see, with this approximation we have recovered our previous results eqs.(3.68)-(3.75) (at leading order).

Now let us examine more carefully the results presented above. For a better understanding of these results it is convenient to plot the energy-density and energy-flux for each of the regions we are studying.

Figures (4.2) and (4.3) show $\langle \hat{T}^{00}(r,t)\rangle$ for r fixed and t fixed respectively. (In all these plots, we are taking a value of $\alpha = 10000$.) We can see that $\langle \hat{T}^{00}(r,t)\rangle$ is always finite. Notice that for large t and r fixed $\langle \hat{T}^{00}(r,t)\rangle$ develops small but negative values (we will discuss this result in the next section). If we fix t, the energy-density is always positive and vanishes asymptotically.

The energy-flux is plotted in figures (4.4)-(4.5). $\langle \hat{T}^{0i}(r,t)\rangle$ also develops small but negative values when r remains fixed and time evolves. It is interesting to notice one aspect in the evolution of the energy-flux as time evolves namely that the energy-flux

always presents very small values, in comparison to the corresponding values for the energy-density (also true for the $t-$fixed and case).

4.3 The total energy of the string

Now, let us compute an analytic expression for the total energy of the string. The total energy is given by the following expression:

$$E_T = \int \langle \hat{T}^{00}(r,t) \rangle dV = 4\pi \int_0^R \langle \hat{T}^{00}(r,t) \rangle r^2 \, dr \tag{4.6}$$

where R acts as a cut-off. From eqs.(3.56) and (4.3) we have:

$$\begin{aligned} E_T &= \frac{2}{\alpha^2 (2\pi)^2} \sqrt{\frac{2\alpha}{\pi}} \int_0^R r \, dr \left[\pi \left(\Phi(\frac{t+r}{\sqrt{2\alpha}}) - \Phi(\frac{t-r}{\sqrt{2\alpha}}) \right) + \right. \\ & \left. \frac{1}{6} \sqrt{\frac{2\pi}{\alpha}} \left((t+r) \, e^{-\frac{(t+r)^2}{2\alpha}} - (t-r) \, e^{-\frac{(t-r)^2}{2\alpha}} \right) \right] . \end{aligned} \tag{4.7}$$

Making the following changes of variables

$$x = t + r, \; y = t - r$$

we can re-write eq.(4.7) as

$$\begin{aligned} E_T &= \frac{2}{\alpha^2 (2\pi)^2} \sqrt{\frac{2\alpha}{\pi}} \left[- \int_t^{t+R} (t-x) \, dx \, \pi \, \Phi(\frac{x}{\sqrt{2\alpha}}) + \int_t^{t-R} (t-y) \, dy \, \pi \, \Phi(\frac{y}{\sqrt{2\alpha}}) \right. \\ & \left. - \frac{1}{6} \sqrt{\frac{2\pi}{\alpha}} \int_t^{t+R} (t-x) \, dx \, x \, e^{-\frac{x^2}{2\alpha}} + \frac{1}{6} \sqrt{\frac{2\pi}{\alpha}} \int_t^{t-R} (t-y) \, dy \, y \, e^{-\frac{y^2}{2\alpha}} \right] . \end{aligned} \tag{4.8}$$

Thus, the total energy can be written in the following form:

$$E_T = \frac{2}{\alpha^2 (2\pi)^2} \sqrt{\frac{2\alpha}{\pi}} \left[P(t+R) - P(t-R) + Q(t+R) - Q(t-R) \right] \tag{4.9}$$

where

$$P(z) = \pi \Phi(\frac{z}{\sqrt{2\alpha}}) \left[\frac{1}{2}(z^2 - \alpha) - tz \right] - \sqrt{2\alpha\pi} \, e^{-\frac{z^2}{2\alpha}} \left(t - \frac{z}{2} \right) \tag{4.10}$$

and

$$Q(z) = \frac{\alpha\pi}{6} \left[\sqrt{\frac{2\alpha}{\pi}} \, \frac{t}{\alpha} \, e^{-\frac{z^2}{2\alpha}} \, (t-z) + \Phi(\frac{z}{\sqrt{2\alpha}}) \right] . \tag{4.11}$$

We can now, take the limits we discussed in chapter 3 but before doing that, let us address our result found in the previous section for the energy-density in the regime $t \to \infty$, r fixed.

Although the notion of a state with negative energy is not known in classical physics this is not the case in quantum field theory. The existence of quantum states with negative energy density is proved to be inevitable as some authors have been able to show [90, 91]. In quantum field theory, the energy density may be negative in a certain space-time region. In order to avoid disturbing effects arising from these negative energy-density regions, we need some constraint on the temporal or spatial extent of the negative energy density as well as on its magnitude. It has been argued [92, 93] that the negative energy $-\Delta E$ localisable in a time of order Δt should satisfy the following quantum inequality:

$$|\Delta E|\,\Delta t \lesssim \hbar. \tag{4.12}$$

The physical implication of this quantum inequality is that an observer cannot see unboundedly large negative energy densities which persist for arbitrarily long periods of time. It is not difficult to see that in the region where the energy density we have computed takes on vanishingly small negative values, the quantum inequality eq.(4.12) is indeed satisfied. Thus, the negative energy density we have found does not play any noticeable physical role as we may have expected.

Working now the asymptotic behaviour for the total energy we obtain

a) $R \gg T$, T large but fixed (remembering that $\alpha \sim T^2$)

$$E_T \sim \left(\frac{R}{T}\right)^2 \frac{1}{T} \tag{4.13}$$

and

b)$R < T$, T large but fixed. From eq.(3.52) we find again:

$$E_T \sim \left(\frac{R}{T}\right)^3 \frac{1}{T}. \tag{4.14}$$

As we can see these results are in total agreement with the ones obtained in chapter 3, as they should be.

4.4 Final Remarks

As we can see from the results obtained throughout this chapter, we have an accurate method to obtain more complete results for the expectation value of the massless sector of the quantum bosonic string energy-momentum tensor in spherical configurations. We have been able to show that this method reproduces the asymptotic behaviour for $\langle \hat{T}^{\mu\nu}(r,t)\rangle$

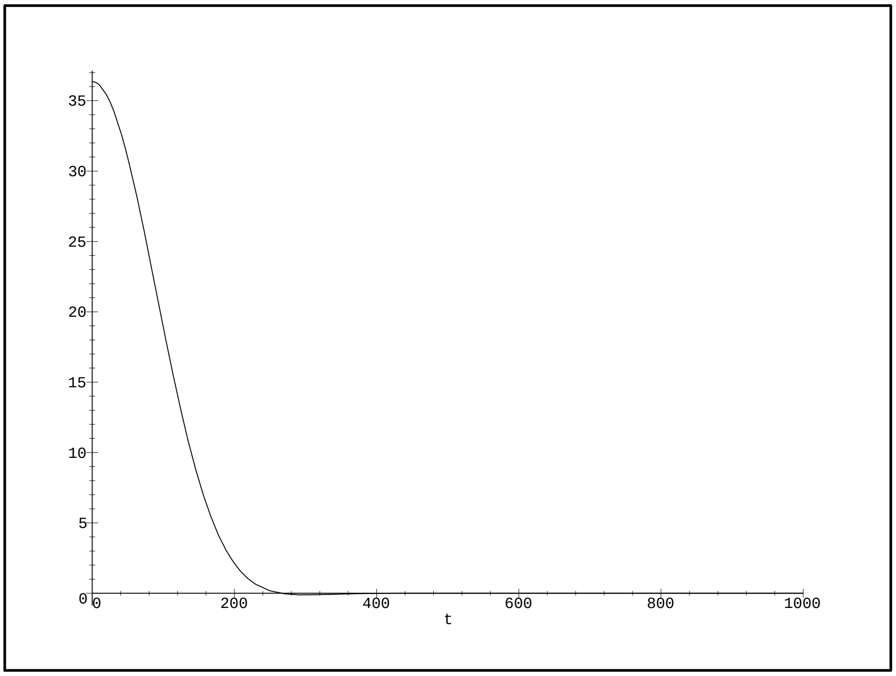

Figure 4.2: $\langle \hat{T}^{00}(r,t) \rangle$ $(\times 10^{-12})$ for r fixed and equal to 0.5, α=10000.

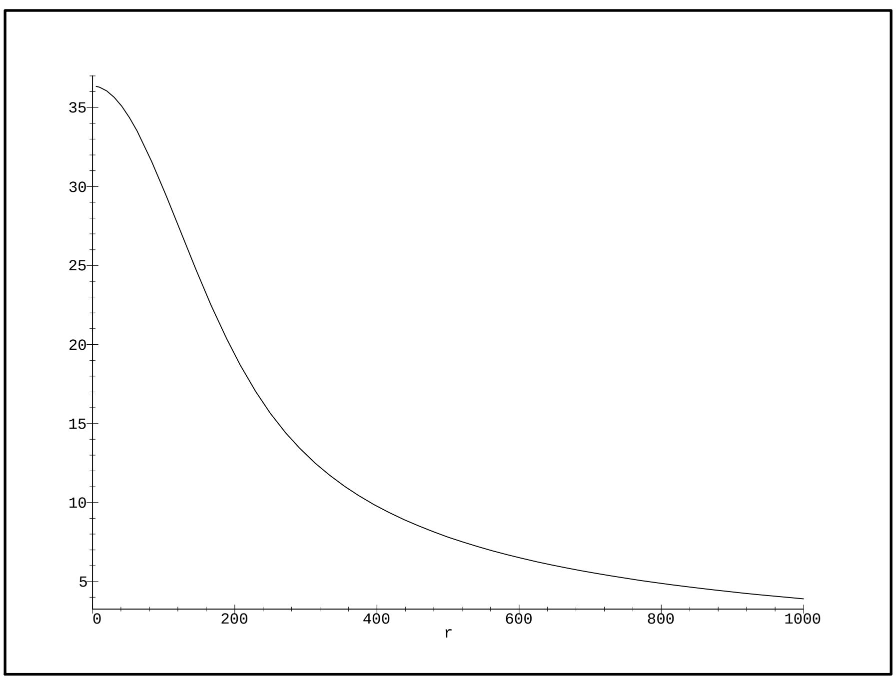

Figure 4.3: $\langle \hat{T}^{00}(r,t) \rangle$ $(\times 10^{-12})$ for t fixed and equal to 0.5, α=10000.

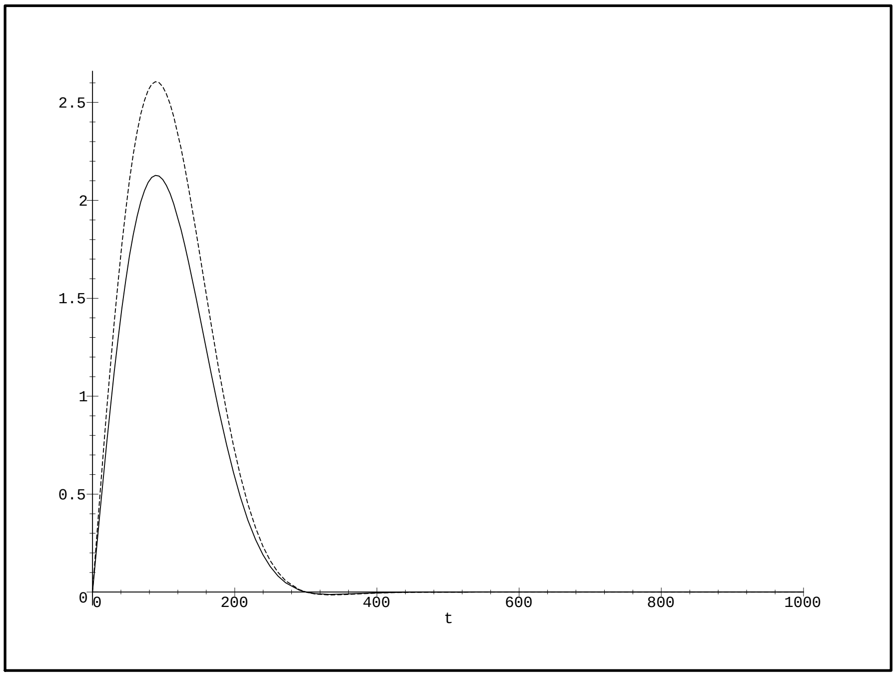

Figure 4.4: $\langle \hat{T}^{01}(r,t) \rangle$ $(\times 10^{-14})$ for r fixed and equal to 0.5, α=10000. Here the solid line shows $\langle \hat{T}^{01}(r,t) \rangle$ in the direction $\phi = \pi/4$, $\gamma = \pi/4$ whereas the dashed line shows $\langle \hat{T}^{01}(r,t) \rangle$ in the direction $\phi = \pi/4$, $\gamma = \pi/3$. The other 2 components of $\langle \hat{T}^{0i}(r,t) \rangle$ behave in a similar fashion, different only from $\langle \hat{T}^{01}(r,t) \rangle$ by a factor introduced by $\hat{x}^i = (\cos\phi \sin\gamma, \sin\phi \sin\gamma, \cos\gamma)$

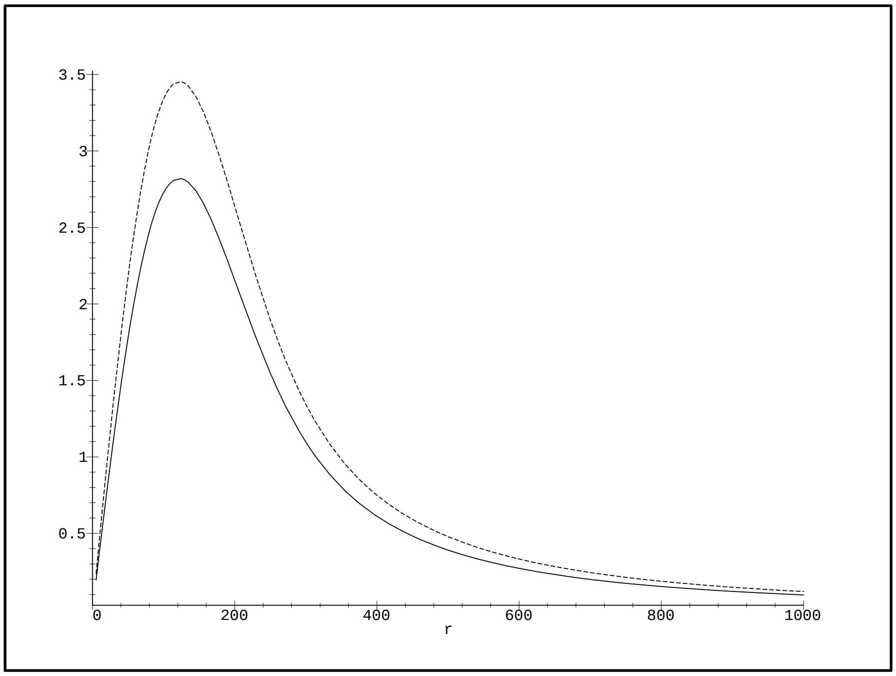

Figure 4.5: $\langle\hat{T}^{01}(r,t)\rangle$ $(\times 10^{-14})$ for t fixed and equal to 0.5, α=10000. Here the solid line shows $\langle\hat{T}^{01}(r,t)\rangle$ in the direction $\phi = \pi/4$, $\gamma = \pi/4$ whereas the dashed line shows $\langle\hat{T}^{01}(r,t)\rangle$ in the direction $\phi = \pi/4$, $\gamma = \pi/3$. The other 2 components of $\langle\hat{T}^{0i}(r,t)\rangle$ behave in a similar fashion, different only from $\langle\hat{T}^{01}(r,t)\rangle$ by a factor introduced by $\hat{x}^i = (\cos\phi\sin\gamma, \sin\phi\sin\gamma, \cos\gamma)$

computed in the previous chapter as well as the asymptotic behaviour of the total energy. We can now use these explicit expression for $\langle \hat{T}^{\mu\nu}(r,t)\rangle$ in order to investigate the weak field limit of Einstein's equations.

CHAPTER 5

The gravitational field of a quantum bosonic string

In this chapter and the next, we will compute and analyse the weak-field metric produced by a quantum bosonic string. We shall see that the gravitational field for a quantum string differs in many ways from that of a *classical string* (cosmic string)[1]. Cosmic strings (amongst other defects) could have been produced as the early Universe went through a phase transition. They have been proposed as a possible explanation for the density fluctuations in the early Universe that may have given rise to the origin of galaxies. Studies of how their gravitational field affects their motion have been presented by a number of authors (see for example: [37, 85, 103]). It has been shown, for example, that the metric around a straight static string is that of a conical space-time; that is, although particles moving near the cosmic string are not affected by any gravitational force from the string, gravitational lensing occurs. This leads to double images with a characteristic separation of about a few seconds of arc [97]. This in principle should lead to observational evidence for the existence of cosmic strings in the early Universe. However, current astronomical observations have not shown yet any evidence in that respect.

Another important effect due to the space-time geometry produced by a cosmic string is the appearance of wakes behind moving strings [105, 106]. This causes particles passing near the string to conglomerate in a region and eventually this region will have a total mass comparable with that of the string itself. These wakes may provide us with an explanation for the observed large scale structure of the Universe.

In this calculation, we will be working in a *far field* approximation to Einstein's equations, and will show that the solution for the gravitational field of a quantum string will hold in regions far from the source. The metric of a quantum string approaches that of a flat Minkowski space-time as we go farther away from the source. This is in agreement

[1] Strictly speaking, this is true in the Nambu approximation where the string is consider to have zero width.

with our weak-field approximation assumption. It should be noticed though, that in these calculations we are not considering the string solutions in a fully self-consistent way; we must not forget that, once we obtain the metric generated by the string, the string will depend now on this new metric. We must remember that String Theory is extremely sensitive to the background upon which the string is moving. So if our weak-field approximation of the metric starts to deviate noticeably from the Minkowski metric, our string solution used in the computation of the string energy-momentum tensor will no longer be valid. The results presented in this chapter, therefore, may be used as long as the string solutions can be considered as those for a flat Minkowski space-time which is the case here.

5.1 The weak field approximation to Einstein's field equations

The weak field approximation is a linearised version of Einstein's field equations. It is based upon the assumption that the metric tensor will deviate only slightly from the Minkowski metric [98]:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

where $|h^{\mu\nu}| \ll 1$. In this way we may neglect all terms not linear in $h_{\mu\nu}$ or its derivatives.

In this approximation, Einstein's field equations read:

$$\Box h^{\mu\nu} - \frac{\partial^2 h^\nu_\lambda}{\partial x^\lambda \partial x^\mu} - \frac{\partial^2 h^\mu_\lambda}{\partial x^\lambda \partial x^\nu} + \frac{\partial^2 h^\lambda_\lambda}{\partial x^\mu \partial x^\nu} = -16\pi G S^{\mu\nu}, \tag{5.1}$$

where

$$S^{\mu\nu} = T^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} T^\lambda_\lambda.$$

These equations cannot be solved in a unique way, the reason for this being that we can always perform the following general transformation:

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x),$$

where $\frac{\partial \epsilon^\mu(x)}{\partial x^\nu}$ is at most of the same magnitude as $h^{\mu\nu}$; ϵ^μ are small and arbitrary functions of x.

In this new coordinate system, x'^μ, the metric is given by

$$g'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\lambda}\frac{\partial x'^\nu}{\partial x^\rho} g^{\lambda\rho}$$

or, since $g^{\mu\nu} \simeq \eta^{\mu\nu} - h^{\mu\nu}$:

$$h'^{\mu\nu} = h^{\mu\nu} - \frac{\partial \epsilon^\mu}{\partial x^\lambda}\eta^{\lambda\nu} - \frac{\partial \epsilon^\nu}{\partial x^\rho}\eta^{\rho\mu}.$$

Therefore, if $h^{\mu\nu}$ is a solution,

$$h'^{\mu\nu} = h^{\mu\nu} - \frac{\partial \epsilon^\mu}{\partial x_\nu} - \frac{\partial \epsilon^\nu}{\partial x_\mu}$$

is also a solution.

We can now choose a gauge in order to work in a particular (convenient) coordinate system. Choosing the Harmonic gauge $g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = 0$ (here $\Gamma^\lambda_{\mu\nu}$ are the Christoffel's symbols), we get to a first order:

$$\frac{\partial h^\mu_\nu}{\partial x^\mu} = \frac{1}{2}\frac{\partial h^\mu_\mu}{\partial x^\nu}. \tag{5.2}$$

Notice that if $h^{\mu\nu}$ does not satisfy this relation, we can always make it comply by performing a change of coordinates. In this gauge, the weak field approximation to Einstein's equations reads:

$$\Box h^{\mu\nu}(x) = -16\pi G\left(T^{\mu\nu}(x) - \frac{1}{2}\eta^{\mu\nu}T^\lambda_\lambda(x)\right). \tag{5.3}$$

Here it may be appropriate to mention briefly, as a reminder, that although working with the linearised version of Einstein's field equations presents clear advantages over working with its exact form, we always have to be very careful about the conclusions we reach from this approximation. The exact gravitational field corresponding to the exact solution may deviate significantly from that derived from the linearised theory if the sources of the field move in a different way from what was in the first place assumed. Therefore, the reliability of the results found in the linearised theory will be higher if we have a good knowledge of the behaviour of our source and if the source is not too massive.

Since in our case the source of the gravitational field is a quantum operator ($\hat{T}^{\mu\nu}(x)$), we must modify eq.(5.3) slightly replacing in its RHS $T^{\mu\nu}(x)$ by $\langle\hat{T}^{\mu\nu}(x)\rangle$:

$$\Box h^{\mu\nu}(x) = -16\pi G\left(\langle\hat{T}^{\mu\nu}(x)\rangle - \frac{1}{2}\eta^{\mu\nu}\langle\hat{T}^\lambda_\lambda(x)\rangle\right). \tag{5.4}$$

This is what is known as a semiclassical approximation to quantum gravity.

In order for the field equations to be integrable, the expectation value $\langle\hat{T}^{\mu\nu}(x)\rangle$ must be divergence free [94], that is

$$\langle\hat{T}^{\mu\nu}(x)\rangle_{;\nu} = 0. \tag{5.5}$$

It should be noted that eq.(5.5) is not a consequence of the equations governing the quantum fields but rather a constraint on these quantities [94] for example, the states which are used to form the expectation values (just as we saw happen in chapter 3).

5.2 Far fields

In the last chapter we derived explicit equations for the expectation value of the string energy-momentum tensor. Therefore, we may now proceed to solve eq.(5.3).

As we have already shown in chapter 3, $\langle \hat{T}^{\lambda}_{\lambda}(x) \rangle = 0$. Thus eq.(5.4) simplifies to

$$\Box h^{\mu\nu}(x) = -16\pi G \, \langle \hat{T}^{\mu\nu}(x) \rangle. \tag{5.6}$$

This differential equation can be solved with the help of the retarded potential Green's function [99] leading to

$$h^{\mu\nu} = 4G \int d^3\vec{x}' \, \frac{\langle \hat{T}^{\mu\nu}(\vec{x}', t_{ret}) \rangle}{|\vec{x} - \vec{x}'|}, \tag{5.7}$$

where

$$t_{ret} = t - \frac{|\vec{x} - \vec{x}'|}{c}.$$

Of course, to this solution we can always add the solution of the homogeneous part:

$$\Box h^{\mu\nu} = 0.$$

So we can interpret the two solutions as follows: the solution (5.7) is the gravitational radiation produced by the source, in our case a quantum bosonic string; the solution of the homogeneous equation represents the gravitational radiation coming in from infinity [98].

The retarded time t_{ret} that appears in the argument of the energy-momentum tensor in eq.(5.7) tells us that gravitational effects due to this source propagate at the speed of light.

At the quantum level the string energy-momentum tensor does not behave like a localised source, we cannot blindly use the retarded potential technique to solve Einstein's field equations; therefore, in order to solve eq.(5.7), we need to make some reasonable assumptions about the behaviour of our source. We have learnt for example in the previous chapter that the string energy-momentum tensor vanishes asymptotically for large values of r and t. In this case, a far field approximation seems to be appropriate (see fig.(5.1)), where we would be interested in computing the gravitational field in a region at a distance r far from the matter distribution. That is, in the region where $r \gg R$, where R acts like a cut-off, in that most of the matter would be concentrated within a region of radius R and we will be neglecting any other contributions from outside this region. We may visualise this scenario as if looking at the gravitational field exerted by a planetary system or a

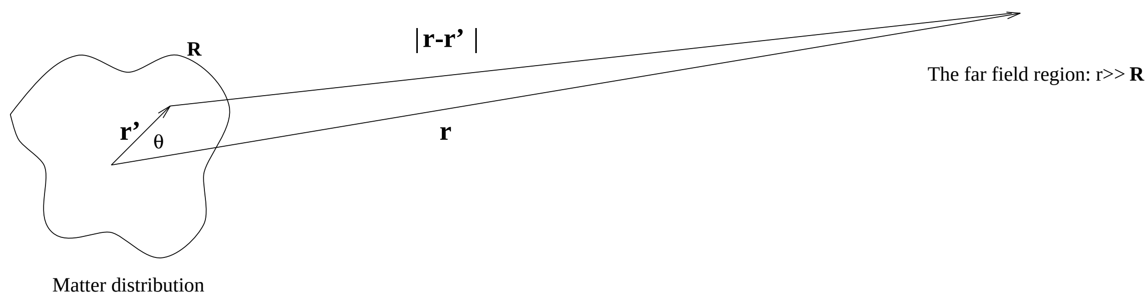

Figure 5.1: The far field approximation. The source is assumed to be contained within a region of radius R. We compute the weak-field metric in a region at a distance r far from the source.

galaxy viewed from a location far away from them and neglecting all interstellar matter in between them and our location. Notice that our calculation for $\langle \hat{T}^{\mu\nu}(\vec{x}, t) \rangle$ was carried out using Gaussian-wave packets peaked around $E = 0$. Since $E = \sqrt{\vec{p}^2 + m^2}$ (because we are always working on-shell) we have that for the massless case we are considering here $E = \sqrt{\vec{p}^2}$ implies that we will be considering the weak gravitational field produced by a source moving with centre of mass velocities small compared with the velocity of light. This kind of approximation is the one presented in most general relativity textbooks (e.g. [94], [95], [98], [102].)

5.3 The quantum bosonic string gravitational field

From chapter 3 eqs.(3.53) and (3.56) we learnt that the quantum string energy-momentum tensor components may be written as:

$$
\begin{aligned}
\langle \hat{T}^{00}(\vec{x}, t) \rangle &= \frac{1}{r} \left[F(t+r) - F(t-r) \right], \\
\langle \hat{T}^{11}(\vec{x}, t) \rangle &= -\frac{1}{r^2} \left[H(t+r) + H(t-r) \right] + \frac{1}{r^3} \left[E(t+r) - E(t-r) \right] + \\
&\quad \cos^2\phi \, \sin^2\gamma \left\{ \frac{1}{r} \left[F(t+r) - F(t-r) \right] + \frac{3}{r^2} \left[H(t+r) + H(t-r) \right] \right. \\
&\quad \left. - \frac{3}{r^3} \left[E(t+r) - E(t-r) \right] \right\}, \\
\langle \hat{T}^{22}(\vec{x}, t) \rangle &= \frac{1}{r^2} \left[H(t+r) + H(t-r) \right] + \frac{1}{r^3} \left[E(t+r) - E(t-r) \right] + \\
&\quad \sin^2\phi \, \sin^2\gamma \left\{ \frac{1}{r} \left[F(t+r) - F(t-r) \right] + \frac{3}{r^2} \left[H(t+r) + H(t-r) \right] \right. \\
&\quad \left. - \frac{3}{r^3} \left[E(t+r) - E(t-r) \right] \right\}, \\
\langle \hat{T}^{33}(\vec{x}, t) \rangle &= \frac{1}{r^2} \left[H(t+r) + H(t-r) \right] + \frac{1}{r^3} \left[E(t+r) - E(t-r) \right] +
\end{aligned}
$$

$$
\begin{aligned}
& \cos^2\gamma \left\{\frac{1}{r}\left[F(t+r)-F(t-r)\right]+\frac{3}{r^2}\left[H(t+r)+H(t-r)\right]\right. \\
& \left. -\frac{3}{r^3}\left[E(t+r)-E(t-r)\right]\right\}, \\
\langle\hat{T}^{01}(\vec{x},t)\rangle \;=\; & -\cos\phi\,\sin\gamma\left\{\frac{1}{r}\left[F(t+r)+F(t-r)\right]+\frac{1}{r^2}\left[H(t+r)-H(t-r)\right]\right\}, \\
\langle\hat{T}^{02}(\vec{x},t)\rangle \;=\; & -\sin\phi\,\sin\gamma\left\{\frac{1}{r}\left[F(t+r)+F(t-r)\right]+\frac{1}{r^2}\left[H(t+r)-H(t-r)\right]\right\}, \\
\langle\hat{T}^{03}(\vec{x},t)\rangle \;=\; & -\cos\gamma\left\{\frac{1}{r}\left[F(t+r)+F(t-r)\right]+\frac{1}{r^2}\left[H(t+r)-H(t-r)\right]\right\}. \qquad (5.8)
\end{aligned}
$$

We will insert expressions (5.8) into eq.(5.7) shortly. First, in order to work out the far field of our quantum string, we need to expand $|\vec{x}-\vec{x}'|$ as a power series [94, 99, 100, 101]:

$$
\begin{aligned}
|\vec{x}-\vec{x}'| &= r-\frac{x^i x'^i}{r}-\frac{1}{2}\frac{x^i x^j}{r^3}\left(x'^i x'^j - r'^2\delta^{ij}\right)+\ldots, \\
\frac{1}{|\vec{x}-\vec{x}'|} &= \sum_{l=0}^{\infty}\frac{r'^l}{r^{l+1}}P_l(cos\theta)=\frac{1}{r}+\frac{x^i x'^i}{r^3}+\frac{1}{2}\frac{x^i x^j}{r^5}\left(3x'^i x'^j - r'^2\delta^{ij}\right)+\ldots \qquad (5.9)
\end{aligned}
$$

where $i,j=1,2,3$. Substituting this expression into the argument $t-|\vec{x}-\vec{x}'|$ of the energy-momentum tensor and expanding the components of $\langle\hat{T}^{\mu\nu}\rangle$, we obtain:

$$
\begin{aligned}
\int d^3\vec{x}'\,\frac{\langle\hat{T}^{\mu\nu}(\vec{x}',t)\rangle}{|\vec{x}-\vec{x}'|} \;=\; & \int d^3\vec{x}'\langle\hat{T}^{\mu\nu}(\vec{x}',t-r)\rangle\left\{\frac{1}{r}+\frac{x^i x'^i}{r^3}+\frac{1}{2}\frac{x^i x^j}{r^5}\left(3x'^i x'^j - r'^2\delta^{ij}\right)+\ldots\right\} \\
& +\int d^3\vec{x}'\langle\dot{\hat{T}}^{\mu\nu}(\vec{x}',t-r)\rangle\left\{\frac{x^i x'^i}{r^2}+\frac{1}{2}\frac{x^i x^j}{r^4}\left(3x'^i x'^j - r'^2\delta^{ij}\right)+\ldots\right\} \\
& +\int d^3\vec{x}'\frac{1}{2}\langle\ddot{\hat{T}}^{\mu\nu}(\vec{x}',t-r)\rangle\frac{x^i x^j x'^i x'^j}{r^3}+\ldots, \qquad (5.10)
\end{aligned}
$$

where the dot means differentiation with respect to time t. The first three terms of this series are the analogue to the 'charge', 'dipole' and 'quadrupole' terms of the electromagnetic field multipole expansion [99].

We will work only with the terms we have displayed above since they are the ones that will dominate the expansion in the region of large r.

Inserting eq.(5.10) into expression (5.7), we obtain:

$$
\begin{aligned}
h^{\mu\nu}(\vec{x},t) \;=\; & \left[h_a^{\mu\nu}(\vec{x},t)+h_b^{\mu\nu}(\vec{x},t)+\frac{1}{2}h_c^{\mu\nu}(\vec{x},t)+\right. \\
& \left. r\dot{h}_b^{\mu\nu}(\vec{x},t)+\frac{r}{2}\dot{h}_c^{\mu\nu}(\vec{x},t)+\frac{1}{2}\ddot{h}_d^{\mu\nu}(\vec{x},t)\right]. \qquad (5.11)
\end{aligned}
$$

where $i=1,2,3$. Here, $h_a^{\mu\nu}(\vec{x},t)$, $h_b^{\mu\nu}(\vec{x},t)$, $h_c^{\mu\nu}(\vec{x},t)$ and $h_d^{\mu\nu}(\vec{x},t)$ are given by:

$$
h_a^{\mu\nu}(\vec{x},t)=\frac{4G}{r}\int d^3\vec{x}'\langle\hat{T}^{\mu\nu}(\vec{x}',t-r)\rangle, \qquad (5.12)
$$

$$
h_b^{\mu\nu}(\vec{x},t)=4G\frac{x^i}{r^3}\int d^3\vec{x}'\,x'^i\langle\hat{T}^{\mu\nu}(\vec{x}',t-r)\rangle, \qquad (5.13)
$$

$$h_c^{\mu\nu}(\vec{x},t) = 4G\frac{x^i x^j}{r^5}\int d^3\,\vec{x}'\;(3\,x'^i x'^j - r'^2\delta^{ij})\;\langle\hat{T}^{\mu\nu}(\vec{x}',t-r)\rangle \tag{5.14}$$

and

$$h_d^{\mu\nu}(\vec{x},t) = 4G\frac{x^i x^j}{r^3}\int d^3\vec{x}'\;x'^i x'^j\langle\hat{T}^{\mu\nu}(\vec{x}',t-r)\rangle. \tag{5.15}$$

Now we can use the explicit expressions eq.(5.8) given in chapter 4 by eqs.(4.3)-(4.5) for the string energy-momentum tensor in order to compute the integrals in eqs.(5.12)-(5.15). Here we have chosen

$$x^i = r(\cos\alpha_0\sin\gamma_0, \sin\alpha_0\sin\gamma_0, \cos\gamma_0)$$

and

$$x'^i = r(\cos\phi\sin\gamma, \sin\phi\sin\gamma, \cos\gamma).$$

After some work (and using the table of integrals in appendix B) we find the following relations between the different components of the metric tensor $h^{\mu\nu}(\vec{x},t)$ (the explicit results for all of the metric components are given in appendix A):

$$\frac{1}{3}h_a^{00}(\vec{x},t) = h_a^{ii}(\vec{x},t), \tag{5.16}$$

$$h_a^{0i}(\vec{x},t) = 0, \tag{5.17}$$

$$h_b^{00}(\vec{x},t) = h_b^{ii}(\vec{x},t) = 0, \tag{5.18}$$

$$h_b^{0i}(\vec{x},t) = \mathcal{W}_i(\alpha_0,\gamma_0)\mathcal{G}_1(\vec{x},t), \tag{5.19}$$

$$h_c^{00}(\vec{x},t) = h_c^{0i}(\vec{x},t) = 0, \tag{5.20}$$

$$h_c^{ii}(\vec{x},t) = \mathcal{Z}_i(\alpha_0,\gamma_0)\mathcal{F}_1(\vec{x},t), \tag{5.21}$$

$$h_d^{ii}(\vec{x},t) = r^2\;\mathcal{S}_i(\alpha_0,\gamma_0)\mathcal{F}_1(\vec{x},t) + \frac{h_d^{00}(\vec{x},t)}{3}, \tag{5.22}$$

$$h_d^{0i}(\vec{x},t) = 0. \tag{5.23}$$

With

$$\begin{aligned} h_a^{00}(\vec{x},t) \quad &= \quad \frac{1}{4\alpha^2\pi\,r}\sqrt{\frac{2\alpha}{\pi}}\;\Bigg\{\Bigg[\Phi(\frac{t-r-R}{\sqrt{2\alpha}}) - \Phi(\frac{t-r+R}{\sqrt{2\alpha}})\Bigg] \\ &\Bigg((t-r)^2 - R^2 + \frac{2\alpha}{3}\Bigg) \\ &+\sqrt{\frac{2\alpha}{\pi}}\left[e^{-\frac{(t-r+R)^2}{2\alpha}} + e^{-\frac{(t-r-R)^2}{2\alpha}}\right]\left(\frac{2R}{3} - t + r\right)\Bigg\}, \end{aligned} \tag{5.24}$$

$$
\begin{aligned}
h_d^{00}(\vec{x},t) &= \frac{2}{3\alpha^2(2\pi)^2\, r}\sqrt{\frac{2\alpha}{\pi}}\ \left\{\frac{\pi}{4}\left[\Phi(\frac{t-r+R}{\sqrt{2\alpha}})-\Phi(\frac{t-r-R}{\sqrt{2\alpha}})\right]\right. \\
&\left(R^4-(t-r)^4-\alpha^2-4\alpha(t-r)^2\right)+\frac{\alpha(t-r)}{4}\sqrt{\frac{2\pi}{\alpha}}\left[e^{-\frac{(t-r+R)^2}{2\alpha}}-e^{-\frac{(t-r-R)^2}{2\alpha}}\right] \\
&\left((t-r)^2-R^2-3\alpha\right)+\frac{\alpha R}{4}\sqrt{\frac{2\pi}{\alpha}}\left[e^{-\frac{(t-r+R)^2}{2\alpha}}+e^{-\frac{(t-r-R)^2}{2\alpha}}\right] \\
&\left.\left((t-r)^2+\frac{R^2}{3}+\alpha\right)\right\}, \qquad (5.25)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}_1(\vec{x},t) &= \frac{\pi}{8\alpha^2(2\pi)^3\, r^3}\sqrt{\frac{2\alpha}{\pi}}\ \left\{\left[\Phi(\frac{t-r+R}{\sqrt{2\alpha}})-\Phi(\frac{t-r-R}{\sqrt{2\alpha}})\right]\times\right. \\
&\left(6(t-r)^2R^2-R^4-5(t-r)^4-5\alpha^2-20\alpha(t-r)^2-4\alpha R^2\right)+ \\
&(t-r)\sqrt{\frac{2\alpha}{\pi}}\left(3(t-r)^2+R^2-3\alpha\right)\left[e^{-\frac{(t-r+R)^2}{2\alpha}}-e^{-\frac{(t-r-R)^2}{2\alpha}}\right]+ \\
&\left.R\sqrt{\frac{2\alpha}{\pi}}\left(9(t-r)^2+\frac{11}{3}R^2+5\alpha\right)\left[e^{-\frac{(t-r+R)^2}{2\alpha}}+e^{-\frac{(t-r-R)^2}{2\alpha}}\right]\right\}, \qquad (5.26)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{G}_1(\vec{x},t) &= \frac{\pi}{2\alpha^2(2\pi)^3\, r^2}\sqrt{\frac{2\alpha}{\pi}}\ \left\{\left[\Phi(\frac{t-r+R}{\sqrt{2\alpha}})-\Phi(\frac{t-r-R}{\sqrt{2\alpha}})\right]\times\right. \\
&\left((t-r)R^2-(t-r)^3-2\alpha(t-r)\right)-\alpha\sqrt{\frac{2\alpha}{\pi}}\left[e^{-\frac{(t-r+R)^2}{2\alpha}}-e^{-\frac{(t-r-R)^2}{2\alpha}}\right]+ \\
&\left.R(t-r)\sqrt{\frac{2\alpha}{\pi}}\left[e^{-\frac{(t-r+R)^2}{2\alpha}}+e^{-\frac{(t-r-R)^2}{2\alpha}}\right]\right\}, \qquad (5.27)
\end{aligned}
$$

and

$$\mathcal{Z}_1(\alpha_0,\gamma_0)=\frac{8\pi}{15}\left(3\cos^2\alpha_0\sin^2\gamma_0-1\right), \qquad (5.28)$$

$$\mathcal{Z}_2(\alpha_0,\gamma_0)=\frac{8\pi}{15}\left(3\sin^2\alpha_0\sin^2\gamma_0-1\right), \qquad (5.29)$$

$$\mathcal{Z}_3(\gamma_0)=\frac{8\pi}{15}\left(2-3\sin^2\gamma_0\right), \qquad (5.30)$$

$$\mathcal{S}_1(\alpha_0,\gamma_0)=\frac{8\pi}{15}\left(\cos^2\alpha_0\sin^2\gamma_0-1\right), \qquad (5.31)$$

$$\mathcal{S}_2(\alpha_0,\gamma_0)=\frac{8\pi}{15}\left(\sin^2\alpha_0\sin^2\gamma_0-1\right), \qquad (5.32)$$

$$\mathcal{S}_3(\gamma_0)=\frac{16\pi}{45}\left(1-\frac{3}{2}\sin^2\gamma_0\right), \qquad (5.33)$$

$$\mathcal{W}_1(\alpha_0,\gamma_0)=\frac{4\pi}{3}\cos\alpha_0\sin\gamma_0, \qquad (5.34)$$

$$\mathcal{W}_2(\alpha_0,\gamma_0)=\frac{4\pi}{3}\sin\alpha_0\sin\gamma_0, \qquad (5.35)$$

$$\mathcal{W}_3(\gamma_0)=\frac{4\pi}{3}\cos\gamma_0, \qquad (5.36)$$

We can see from eqs.(5.16)-(5.36), that all the metric components are written in terms of Gaussian exponentials and error functions. As we will see in the next chapter, all of these terms tend to zero as $r \to \infty$ (t_{ret} fixed). Thus, all the metric components presented here approach those of a flat Minkowski metric at large distances from the matter distribution, as we should have expected.

We must not forget that in this chapter we have stated that most of the mass distribution is localised inside a region of radius R and we are neglecting all the contribution from matter outside this region. The calculations in this chapter make sense as long as we are far away from this distribution; therefore, in principle our calculations do not depend on how large R is as long as $r \gg R$.

With the results we have just obtained, we can now write the metric for a massless quantum bosonic string as:

$$\begin{aligned}
dS^2 \;=\; & \left(1 + h_a^{00} + \frac{1}{2}\ddot{h}_d^{00}\right) dt^2 \\
& - \left(1 - \frac{1}{3}h_a^{00} - \frac{1}{2}\mathcal{Z}_1(\alpha_0,\gamma_0)\mathcal{F}_1(\vec{x},t) - \frac{r}{2}\mathcal{Z}_1(\alpha_0,\gamma_0)\dot{\mathcal{F}}_1(\vec{x},t)\right. \\
& \left. - \frac{r^2}{2}\mathcal{S}_1(\alpha_0,\gamma_0)\ddot{\mathcal{F}}_1(\vec{x},t) - \frac{1}{6}\ddot{h}_d^{00}\right) dx^2 \\
& - \left(1 - \frac{1}{3}h_a^{00} - \frac{1}{2}\mathcal{Z}_2(\alpha_0,\gamma_0)\mathcal{F}_1(\vec{x},t) - \frac{r}{2}\mathcal{Z}_2(\alpha_0,\gamma_0)\dot{\mathcal{F}}_1(\vec{x},t)\right. \\
& \left. - \frac{r^2}{2}\mathcal{S}_2(\alpha_0,\gamma_0)\ddot{\mathcal{F}}_1(\vec{x},t) - \frac{1}{6}\ddot{h}_d^{00}\right) dy^2 \\
& - \left(1 - \frac{1}{3}h_a^{00} - \frac{1}{2}\mathcal{Z}_3(\gamma_0)\mathcal{F}_1(\vec{x},t) - \frac{r}{2}\mathcal{Z}_3(\gamma_0)\dot{\mathcal{F}}_1(\vec{x},t)\right. \\
& \left. - \frac{r^2}{2}\mathcal{S}_3(\gamma_0)\ddot{\mathcal{F}}_1(\vec{x},t) - \frac{1}{6}\ddot{h}_d^{00}\right) dz^2 + \\
& \left(\mathcal{W}_1(\alpha_0,\gamma_0)\mathcal{G}_1(\vec{x},t) + r\mathcal{W}_1(\alpha_0,\gamma_0)\dot{\mathcal{G}}_1(\vec{x},t)\right) dxdt + \\
& \left(\mathcal{W}_2(\alpha_0,\gamma_0)\mathcal{G}_1(\vec{x},t) + r\mathcal{W}_2(\alpha_0,\gamma_0)\dot{\mathcal{G}}_1(\vec{x},t)\right) dydt + \\
& \left(\mathcal{W}_3(\gamma_0)\mathcal{G}_1(\vec{x},t) + r\mathcal{W}_1(\gamma_0)\dot{\mathcal{G}}_1(\vec{x},t)\right) dzdt. \qquad (5.37)
\end{aligned}$$

We can see that this metric presents anisotropies introduced by the functions $\mathcal{Z}_i(\alpha_0,\gamma_0)$, $\mathcal{S}_i(\alpha_0,\gamma_0)$, $\mathcal{W}_i(\alpha_0,\gamma_0)$.

In the following chapter we will discuss in a more ‘graphical’ way the properties of the far field approximation we have developed here and we will mention some results of the gravitational field of classical strings (cosmic strings).

CHAPTER 6

The properties of the far field of a quantum bosonic string

In the previous chapter we dealt with most of the technical issues involved in computing the weak-field metric produced by a quantum bosonic string. Here we will focus on the analysis of the results found there. First we must outline the main differences between our present computations and those that can be found in the literature regarding cosmic strings (classical strings).

As we have said elsewhere, cosmic strings are, from the mathematical point of view, nothing more than very long classical strings (provided we ignore their microstructure). Saying that cosmic strings are classical strings means that in such a theory we will not have to worry about anomalies since these only emerge when we quantise the theory.

It is important to notice that we cannot make a direct comparison between the results presented in this chapter and the results for cosmic strings. Physically, quantum fundamental strings are very different objects from cosmic strings. In addition we have that for example the solutions found by Vilenkin [97] for static cosmic strings, are not compatible with our computation since fundamental strings are not static (in general they move at relativistic speeds). Therefore, it should not surprise us that our results present a different type of behaviour for the $h^{\mu\nu}(\vec{x}, t)$ metric from those found for cosmic strings.

6.1 The quantum string dipole and quadrupole radiation

As was seen in chapter 5, the far field approximation consists of a multipole expansion in terms of the 'field' $h^{\mu\nu}$ and its derivatives with respect to time. This expansion is in clear analogy with the one studied in the electro-magnetic theory [94, 99]. In both cases the main contributions to the multipole expansion come from the first three terms of the series; that is: the 'charge' term, the 'dipole moment' term and the 'quadrupole moment' term.

6.1.1 The dipole radiation

First of all let us check the well established result that gravitational dipole radiation is zero when the energy-momentum tensor in the source is a conserved quantity [94, 95, 98]. That is, we have to check that there is no radiation emerging from the *angular momentum*. The angular momentum can be defined as follows [94]:

$$\Omega^i = \varepsilon_k^{ij} \int d^3\vec{x}' x'^k \langle \hat{T}_{0j}(\vec{x}', t-r)\rangle, \tag{6.1}$$

where ε_k^{ij} is the 3-dimensional antisymmetric tensor. Thus, from eq.(6.1) we find:

$$\begin{aligned}
\Omega^1 &= \varepsilon_k^{1j} \int d^3\vec{x}' x'^k \langle \hat{T}_{0j}(\vec{x}', t-r)\rangle, \\
\Omega^1 &= \varepsilon_k^{12} \int d^3\vec{x}' x'^k \langle \hat{T}_{02}(\vec{x}', t-r)\rangle + \varepsilon_k^{13} \int d^3\vec{x} x'^k \langle \hat{T}_{03}(\vec{x}', t-r)\rangle, \\
\Omega^1 &= \varepsilon_3^{12} \int d^3\vec{x} x'^3 \langle \hat{T}_{02}(\vec{x}', t-r)\rangle + \varepsilon_2^{13} \int d^3\vec{x} x'^2 \langle \hat{T}_{03}(\vec{x}', t-r)\rangle.
\end{aligned}$$

Now, substituting for $\langle \hat{T}_{02}\rangle$ and $\langle \hat{T}_{03}\rangle$ from eq.(5.8):

$$\begin{aligned}
\Omega^1 = & -\varepsilon_3^{12} \int dr' r'^3 \cos\gamma \sin\phi \sin^2\gamma \, d\gamma \, d\phi \left\{ \frac{1}{r'} \left[F(t-r+r') + F(t-r-r')\right] + \right. \\
& \left. \frac{1}{r'^3} \left[H(t-r+r') - H(t-r-r')\right] \right\} - \\
& \varepsilon_2^{13} \int dr' r'^3 \cos\gamma \sin\phi \sin^2\gamma \, d\gamma \, d\phi \left\{ \frac{1}{r'} \left[F(t-r+r') + F(t-r-r')\right] + \right. \\
& \left. \frac{1}{r'^3} \left[H(t-r+r') - H(t-r-r')\right] \right\},
\end{aligned} \tag{6.2}$$

hence,

$$\Omega^1 = 0. \tag{6.3}$$

Similarly we find that $\Omega^2 = \Omega^3 = 0$. Thus, since there is no angular momentum here, we may conclude that there is no gravitational radiation radiated in the form of dipole radiation (we recall that radiation is associated with the change in momentum with respect to time). Indeed the dipole radiation must be zero because, in general relativity the angular momentum of the system is a constant of motion the general result being: $\Omega^i = constant$, thus there cannot possibly be any radiation emerging from the angular momentum.

6.1.2 The quadrupole radiation

Let us investigate now the behaviour of the quadrupole radiation of the quantum bosonic string. The following calculation is an easy way to obtain an accurate estimate of the quadrupole gravitational radiation without going into long tensor calculus [95].

The quadrupole moment can be defined as [94, 99, 101, 102]:

$$Q^{ij} = \int d^3x' \; T^{00}(\vec{x}', t-r) \; x'^i x'^j \tag{6.4}$$

or as

$$Q^{ij}_{red} = \int d^3x' \; T^{00}(\vec{x}', t-r) \; \left(3x'^i x'^j - r'^2 \delta^{ij}\right) \tag{6.5}$$

In our calculation $T^{00}(\vec{x}', t-r)$ must be replaced of course by $\langle \hat{T}^{00}(\vec{x}', t-r) \rangle$. Eq.(6.5) is known as the *reduced* quadrupole moment and it is useful when one chooses to work in the transverse-traceless gauge for the weak-field metric since Q^{ij}_{red} is a traceless tensor. An examination of eq.(6.5) shows that this tensor is of little use to us in order to determine the quadrupole radiation since it identically vanishes. In any case since we have not chosen to work in the transverse-traceless gauge we must work with eq.(6.4) in order to compute the radiation associated with it.

Let us consider the following two expressions arising from energy-momentum conservation.

$$\frac{\partial \langle \hat{T}_{i0} \rangle}{\partial t} - \frac{\partial \langle \hat{T}_{ij} \rangle}{\partial x^j} = 0, \tag{6.6}$$

$$\frac{\partial \langle \hat{T}_{00} \rangle}{\partial t} - \frac{\partial \langle \hat{T}_{0j} \rangle}{\partial x^j} = 0. \tag{6.7}$$

Let us multiply eq.(6.6) by x^k:

$$\frac{\partial \langle \hat{T}_{i0} \rangle}{\partial t} \, x^k - \frac{\partial \langle \hat{T}_{ij} \rangle}{\partial x^j} \, x^k = 0, \tag{6.8}$$

then integrating this expression over a volume we obtain:

$$\int dV \; \frac{\partial \langle \hat{T}_{i0} \rangle}{\partial t} \, x^k = \int dV \; \frac{\partial (\langle \hat{T}_{ij} \rangle \, x^k)}{\partial x^j} - \int dV \; \langle \hat{T}_i^k \rangle. \tag{6.9}$$

Since $\langle \hat{T}^{ij} \rangle \to 0$ at infinity we have that the first integral on the RHS of eq.(6.9) can be dropped (after using Gauss theorem), thus:

$$\begin{aligned} \int dV \; \frac{\partial \langle \hat{T}^{i0} \rangle}{\partial t} \, x^k &= -\int dV \; \langle \hat{T}^{ik} \rangle \\ \int dV \; \langle \hat{T}^{ik} \rangle &= -\frac{1}{2} \frac{\partial}{\partial t} \int dV \; \left(\langle \hat{T}^{i0} \rangle \, x^k + \langle \hat{T}^{k0} \rangle \, x^i \right). \end{aligned} \tag{6.10}$$

Now, let us consider eq.(6.7). Multiplying this expression by $x^i x^k$ we obtain:

$$\frac{\partial \langle \hat{T}_{00} \rangle}{\partial t} \, x^i x^k - \frac{\partial \langle \hat{T}_{0j} \rangle}{\partial x^j} \, x^i x^k = 0. \tag{6.11}$$

Integrating this expression over a volume we find

$$\int dV \, \frac{\partial \langle \hat{T}^{00} \rangle}{\partial t} \, x^i x^k \;\; = \;\; -\int dV \, \frac{\partial}{\partial x^j} \left(\langle \hat{T}^{0j} \rangle \, x^i x^k \right) \\ -\int dV \, (\langle \hat{T}^{0i} \rangle \, x^k + \langle \hat{T}^{0k} \rangle \, x^i), \tag{6.12}$$

Using Gauss theorem again we can drop the first of the integrals in the RHS of eq.(6.12):

$$\int dV \, \frac{\partial \langle \hat{T}^{00} \rangle}{\partial t} \, x^i x^k \;\; = \;\; -\int dV \, (\langle \hat{T}^{0i} \rangle \, x^k + \langle \hat{T}^{0k} \rangle \, x^i). \tag{6.13}$$

Comparing eq.(6.13) with eq.(6.10) we can re-write eq.(6.13) as

$$\frac{\partial^2}{\partial t^2} \int dV \, \langle \hat{T}^{00} \rangle \, x^i x^k = 2 \int dV \, \langle \hat{T}^{ik} \rangle \tag{6.14}$$

thus, we have that

$$\int dV \, \langle \hat{T}^{ik} \rangle \;\; = \;\; \frac{1}{2} \ddot{Q}^{ik}. \tag{6.15}$$

At this point it may be worth to remind the reader what the different components of the energy-momentum tensor actually represent [100, 102]:

$T^{00} =$ energy density,

$T^{0i} =$ energy flux across x^i surface,

$T^{ij} =$ flux of i momentum across j surface.

Thus, the LHS of eq.(6.15) represents the i-momentum which has crossed a surface perpendicular to the k-axis. The radiation is given by the change of momentum crossing the surface with respect to time:

$$\mathcal{P}^{ik} \sim \dddot{Q}^{ik} \tag{6.16}$$

This is the power flowing from one side of our system to another. The power output (*luminosity*) is given approximately by the square of $\mathcal{P}^{ik}$ [95]:

$$\mathcal{I} \sim \dddot{Q}^{ik} \dddot{Q}_{ik} \tag{6.17}$$

this expression as it stands is a good estimate for the order of magnitude of the *luminosity* of the gravitational radiation.

We can now compute $\dddot{Q}^{ik}$ from the explicit expressions (5.8) for the string energy-momentum tensor obtaining the following results:

$$\dddot{Q}^{jl} = 0, \quad j \neq l \tag{6.18}$$

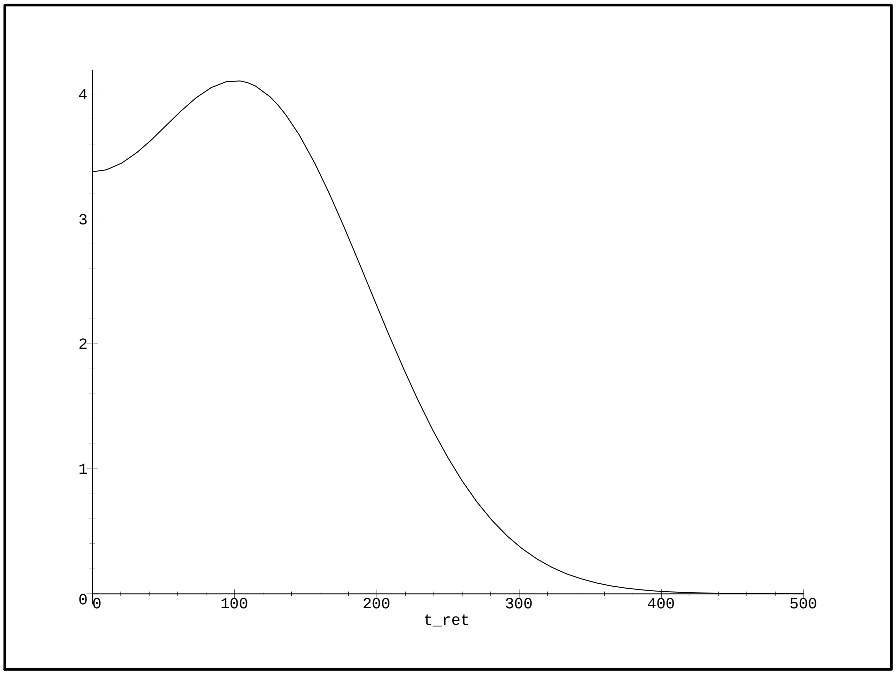

Figure 6.1: The $\dddot{Q}^{ik}$ quadrupole power flux of radiation for a quantum bosonic string in the far field region, $\alpha = 10000$. ($\times 10^{-6}$)

$$\begin{aligned}
\dddot{Q}^{ii}(\vec{x}, t) \quad = \quad & \frac{1}{6\alpha^2\pi^2} \left\{ \left[e^{-\frac{(t-r-R)^2}{2\alpha}} - e^{-\frac{(t-r+R)^2}{2\alpha}} \right] \times \right. \\
& \left(\frac{2\alpha}{3} - \frac{R^2}{3} + (t-r)^2 - R(t-r) \right) + \\
& \left[e^{-\frac{(t-r+R)^2}{2\alpha}} + e^{-\frac{(t-r-R)^2}{2\alpha}} \right] \left((t-r)^2 - \frac{2R}{3}(t-r) + \alpha \right) + \\
& \left. 2(t-r) \left[\Phi(\frac{t-r-R}{\sqrt{2\alpha}}) - \Phi(\frac{t-r+R}{\sqrt{2\alpha}}) \right] \right\}. \qquad (6.19)
\end{aligned}$$

The $\dddot{Q}^{ik}$ and $\dddot{Q}^{ik}\,\dddot{Q}_{ik}$ terms are shown in figures (6.1)-(6.2). We notice that all the $\dddot{Q}^{ik}$ components as well as $\dddot{Q}^{ik}\,\dddot{Q}_{ik}$ are finite in the far field region and that they vanish asymptotically as $t_{ret} \to \infty$. The peak in these figures occurs at the elapsed time for the gravitational radiation to damp if, for example, turbulence, heat conduction or any other effect do not damp it sooner [95].

In summary we have learnt in this section that quantum bosonic strings produce no dipole radiation, but do radiate gravitationally in the form of quadrupole radiation. This general picture agrees with standard results of general relativity regarding the nature of gravitational radiation.

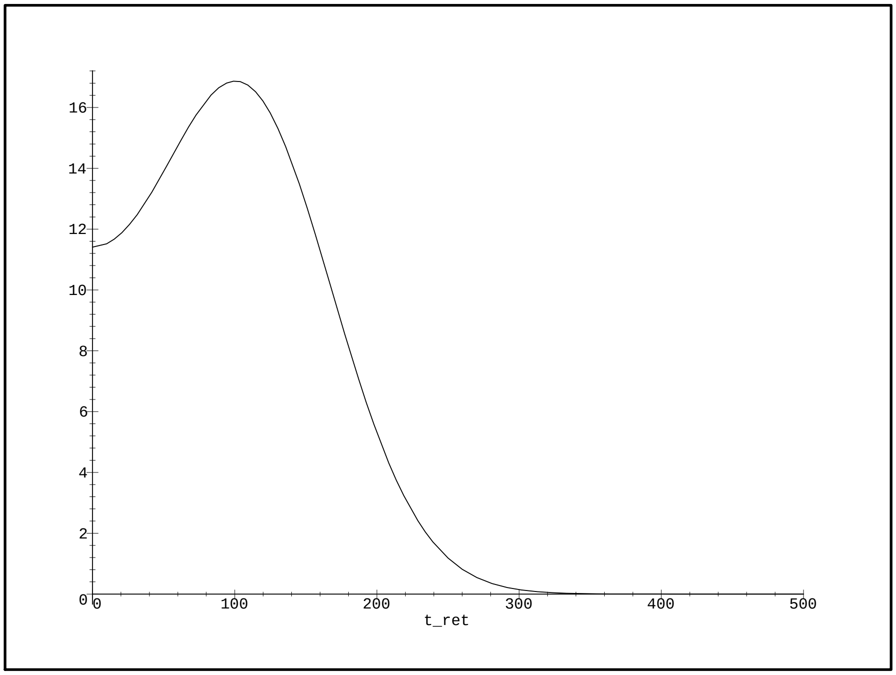

Figure 6.2: The $\dddot{Q}^{ik}\,\dddot{Q}_{ik}$ quadrupole power output of radiation for a quantum bosonic string in the far field region, $\alpha = 10000$. $(\times 10^{-12})$

6.2 The quantum string gravitational field in the far field approximation

So far we have been discussing the gravitational radiation properties of a quantum string. Now let us discuss another important aspect: the space-time geometry induced by a quantum string (the weak field metric components are fully presented in appendix A). One of the first important things we should notice is that our weak-field approximation does not breakdown in the region under study, which is the region $r \gg R$. From here onwards the results stated in the remainder of the chapter are valid only in this far field region. From these results and our discussion in the previous sections, it is expected that individual massless quantum strings produce very small disturbances in the space-time.

We can see from the expression for the first term of our multipole expansion (eq.(5.12)) that this term is finite in the region where r is large. This term is the actual gravitational field produced by the string in the absence of higher order terms in the multipole expansion of the gravitational field worked out in the previous chapter.

Let us start by plotting this term, $h_a^{00}(\vec{x}, t)$ fixing the time t_{ret} (see fig.(6.3)). We see that the gravitational field of a quantum bosonic string without the contribution from the

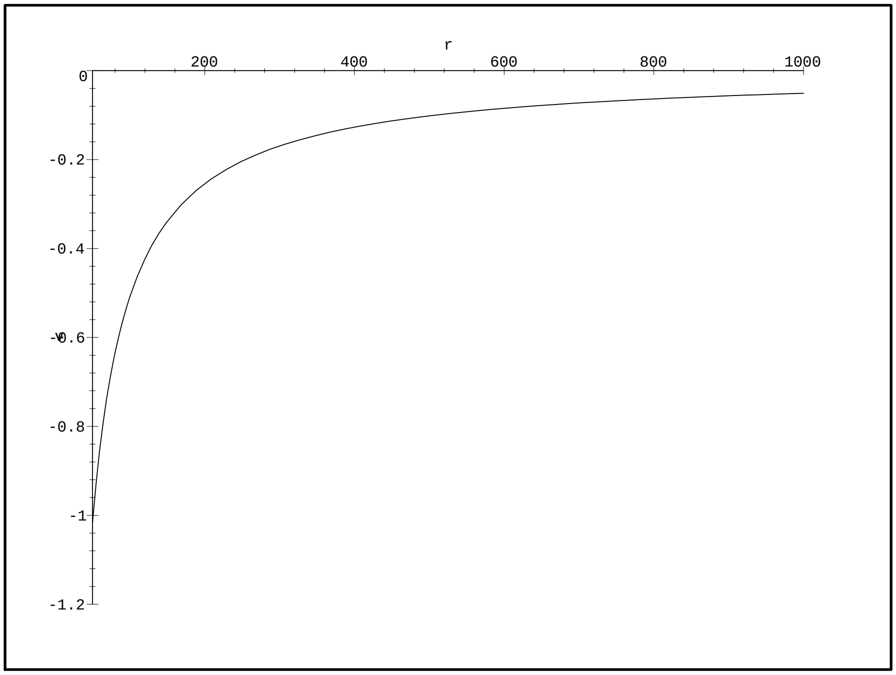

Figure 6.3: A plot of $h_a^{00}(r)$. This term represents the gravitational field induced by the matter source. ($r \gg R$, t_{ret} fixed and $\alpha = 10000$. In units of $G \times 10^{-6}$.)

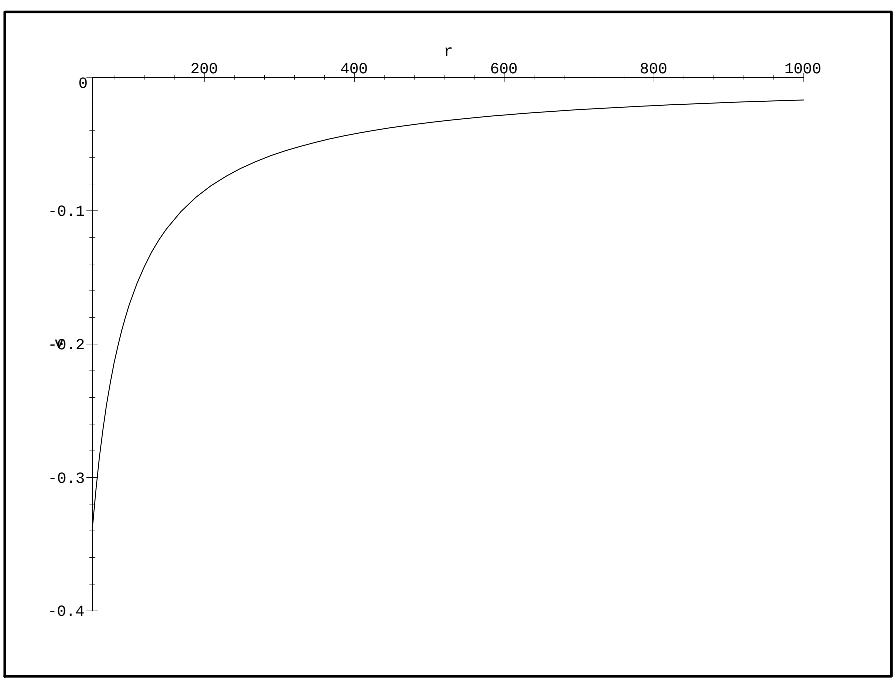

Figure 6.4: A plot of $h_a^{ii}(r)$. ($r \gg R$, t_{ret} fixed and $\alpha = 10000$. In units of $G \times 10^{-6}$.)

gravitational radiation is asymptotically flat. The behaviour of the space elements $h_a^{ii}(r)$ of the metric is completely similar to the one for $h_a^{00}(r)$ as we can see from eq.(5.16) and figure (6.4). In these plots we have used the following set of values for α, R and t_{ret}: ($\alpha = 10^4$, $R = 0.5$,$t_{ret} = 1$.)

Now, let us plot the second order term components of the metric eq.(5.13) (see fig.(6.5)). As we can see from eqs.(5.18)-(5.19), only the h_b^{0i} component of the metric contributes at second order to $h^{\mu\nu}$.

For a fixed t_{ret} we can estimate how strongly the higher order terms contribute to the metric by estimating the ratio $h_a^{\mu\nu}/h_{h.o.}^{\mu\nu} \sim h_a^{00}/h_{h.o.}^{\mu\nu}$ where by $h_{h.o.}^{\mu\nu}$ we mean the higher order terms of the metric in eq.(5.11). From eqs.(5.16)-(5.36) we obtain:

$$\frac{h_a^{00}}{h_b^{0i}} \sim \frac{\alpha\, r}{R\, t_{ret}^2}. \tag{6.20}$$

We can see from this expression that the contribution of h_b^{0i} to the metric is negligible compared to the contribution given by the 'monopole' or 'charge' terms whenever r or α are large (r large is always the case in this far field approximation) and gets smaller as $r \to \infty$ or $\alpha \to \infty$. We can see in fig.(6.5) that for the given values of α, R, and t_{ret} we have used to plot the metric components the contribution of h_b^{0i} to the metric is about 8 orders of magnitude smaller than the 'monopole' terms shown in figures (6.3) and (6.4). Notice that if this term or any of the other higher order terms in our multipole expansion of the gravitational field were to make the space-time metric deviate significantly from that of a Minkowski space-time, we would need to take into account the curvature on the metric induced by this term when we derive the string equations of motion in the first place. We would expect in that hypothetical case that the behaviour of the string equations of motion would change drastically from the one we have been using in this thesis and therefore the string solutions we have used in this computation would no longer be valid. The validity of our results therefore, would have been encompassed in a region $r_0 > r \gg R$, where r_0 would be the critical point in which our original quantum string solutions need to be modified in order to take into account the curvature of the space-time.

We now turn our attention to the contribution from the $h_c^{\mu\nu}$ term eq.(5.14). We have seen in eq.(5.20) that there is no 'mass'-quadrupole term h_c^{00} contributing to the gravitational field. In fact from the same eq.(5.20) we can see that the only contributions to the metric given by the $h_c^{\mu\nu}$ term are through its space components. This term is plotted in

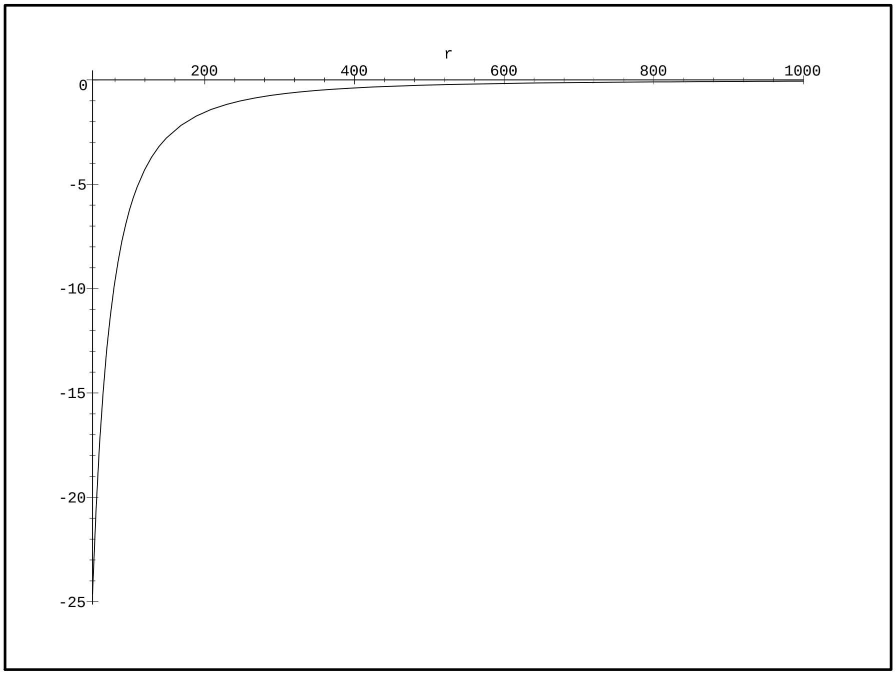

Figure 6.5: Behaviour of the term $h_b^{0i}(r)$ of the metric. ($r \gg R$, t_{ret} fixed and $\alpha = 10000$. In units of $G \times 10^{-14}$.)

fig.(6.6). The ratio $h_a^{00}/h_{h.o.}^{\mu\nu}$ for this term is given by

$$\frac{h_a^{00}}{h_c^{ii}} \sim \frac{r^2}{R\, t_{ret}}. \tag{6.21}$$

We can see that h_c^{ii} makes a minimal contribution to the metric (as compared to h_a^{00}) when r is large. We can see in fig.(6.6) that for the given values of α, R, and t_{ret} we have used to plot the metric components its contribution is about 5 orders of magnitude smaller than the 'monopole' term. Similarly, we can estimate the ratio $h_a^{00}/h_{h.o.}^{\mu\nu}$ for all the other terms that contribute to the metric produced by a massless quantum bosonic string. these terms involve the time derivatives of $h_b^{0i}(\vec{x},t)$, $h_c^{ii}(\vec{x},t)$, $h_d^{00}(\vec{x},t)$ and $h_d^{ii}(\vec{x},t)$ in eq.(5.11). Their ratios $h_a^{00}/h_{h.o.}^{\mu\nu}$ are given by

$$\frac{h_a^{00}}{r\dot{h}_b^{0i}} \sim \frac{\alpha}{R\, t_{ret}}, \tag{6.22}$$

$$\frac{h_a^{00}}{r/2\dot{h}_c^{ii}} \sim \frac{r}{R}, \tag{6.23}$$

$$\frac{h_a^{00}}{1/2\ddot{h}_d^{00}} \sim \frac{\alpha}{R\, t_{ret}} \tag{6.24}$$

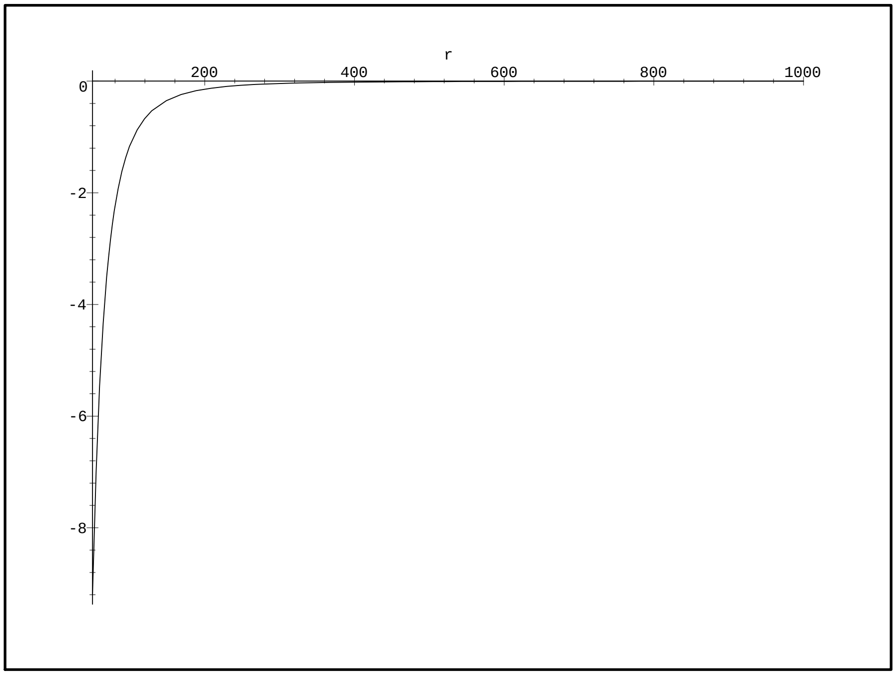

Figure 6.6: The quadrupole term $h_c^{ii}(r)$ of the metric. ($r \gg R$, t_{ret} fixed and $\alpha = 10000$. In units of $G \times 10^{-11}$.)

and

$$\frac{h_a^{00}}{1/2\ddot{h}_d^{ii}} \sim \frac{\alpha}{R\, t_{ret}}. \tag{6.25}$$

We can see from expressions (6.22)-(6.25) that the contributions from these higher order terms in the metric are also small compared to the ‘monopole’ term whenever r or α are large. For the set of values we are using to plot our results we find that the $\dot{h}_b^{0i}$ terms give a contribution about 5 orders of magnitude smaller than the 'monopole' term and $\dot{h}_c^{ii}$ a contribution 3 orders of magnitude smaller than the 'monopole' term. The contribution from the $\ddot{h}_d^{00}$ is also small, their contribution to the metric being about 4 orders of magnitude smaller than the 'monopole' term. Finally, the $\ddot{h}_d^{ii}$ term is also four orders of magnitude smaller than its corresponding 'monopole' term (see figs.(6.7)-(6.10).)

6.3 The gravitational force exerted on non-relativistic particles by a quantum string

We will proceed now to study the properties of the gravitational field $h^{00}(\vec{x}, t)$ derived in the previous chapter. Let us start by considering the force non-relativistic particles would

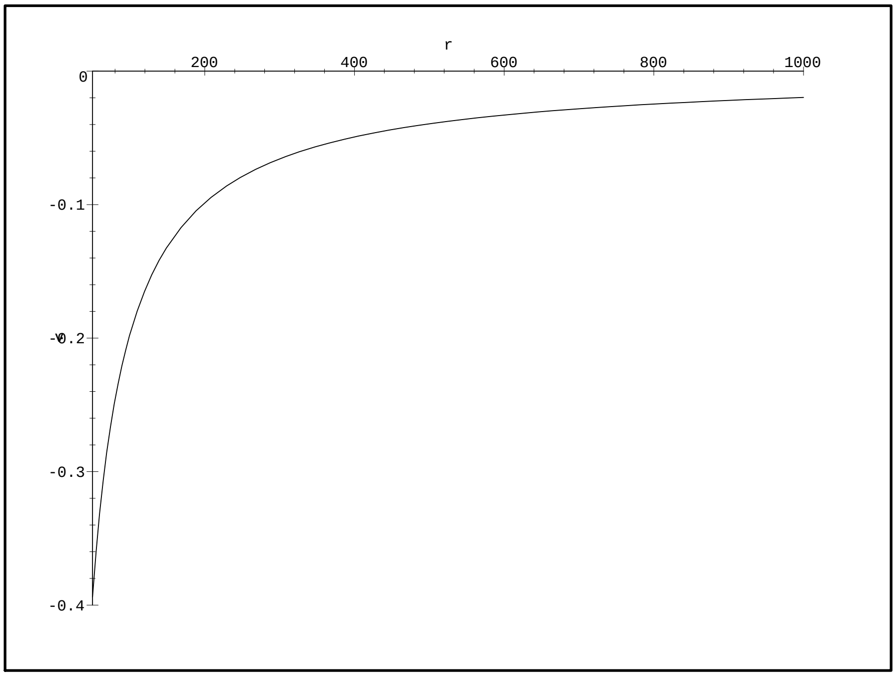

Figure 6.7: The $\dot{h}_b^{0i}(r)$ component of the metric. ($r \gg R$, t_{ret} fixed and $\alpha = 10000$. In units of $G \times 10^{-10}$.)

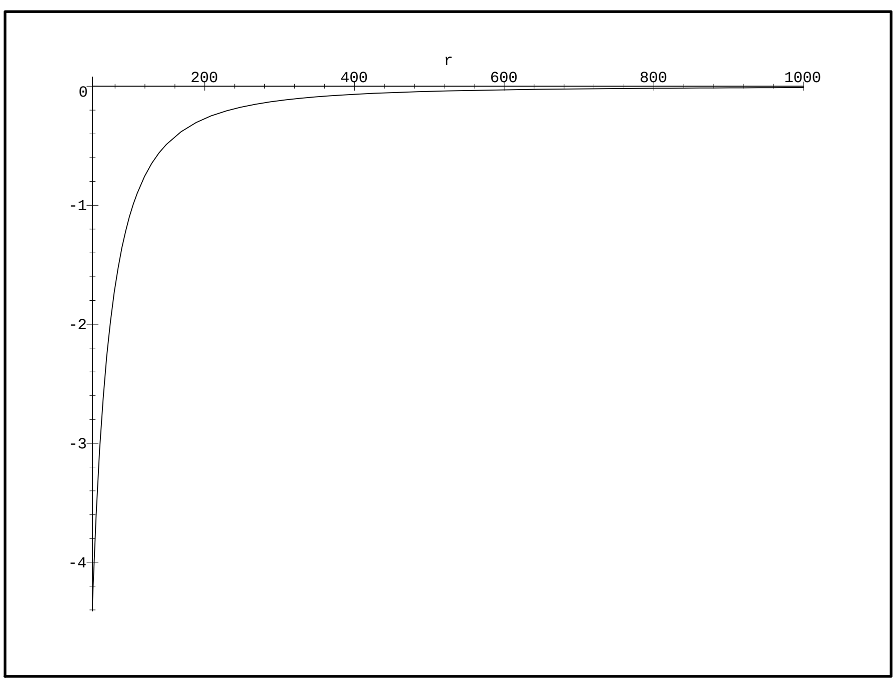

Figure 6.8: The $\dot{h}_c^{ii}(r)$ component of the metric. ($r \gg R$, t_{ret} fixed and $\alpha = 10000$. In units of $G \times 10^{-9}$.)

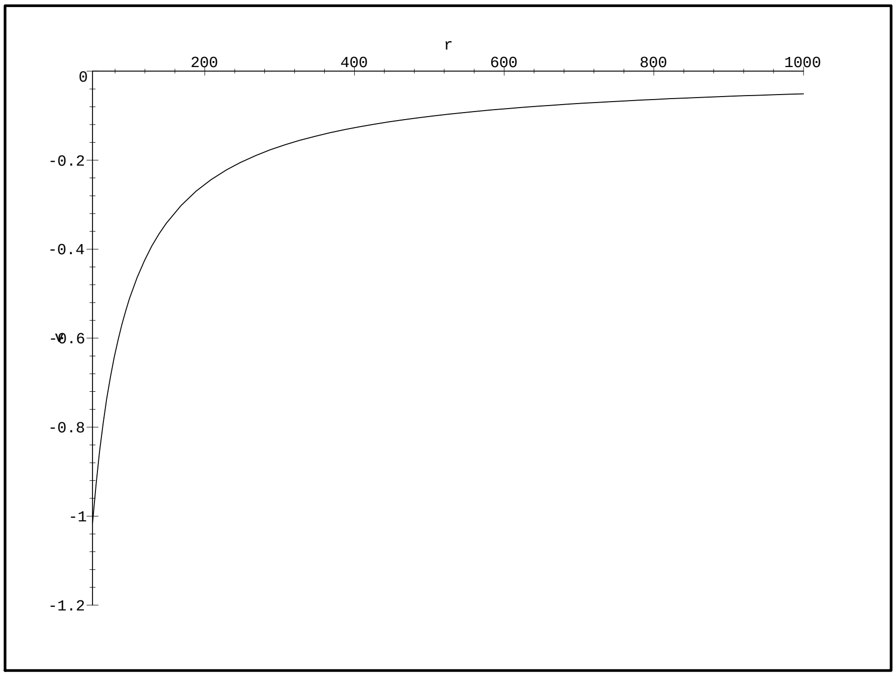

Figure 6.9: The $\ddot{h}_d^{00}(r)$ component of the metric. ($r \gg R$, t_{ret} fixed and $\alpha = 10000$. In units of $G \times 10^{-10}$.)

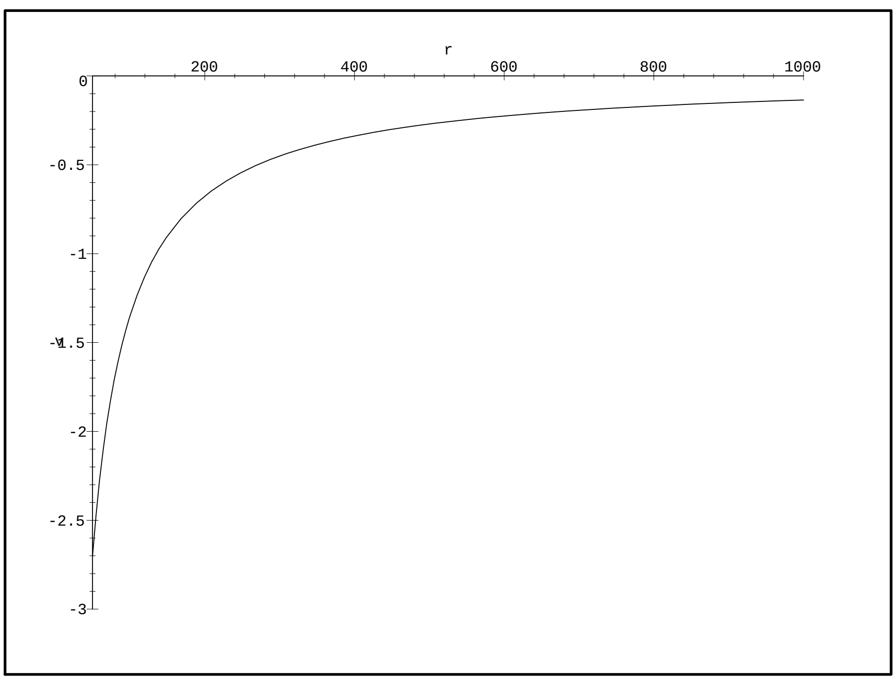

Figure 6.10: The $\ddot{h}_d^{ii}(r)$ component of the metric. ($r \gg R$, t_{ret} fixed and $\alpha = 10000$. In units of $G \times 10^{-11}$.)

experience if they were to move in such a gravitational field.

The force exerted on a non-relativistic particle by $h^{00}(\vec{x},t)$ is given by [95, 98]

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{00} = 0, \tag{6.26}$$

where

$$\Gamma^i_{00} \;=\; -\frac{1}{2}(2h_{0i,0} - h_{00,i}) \tag{6.27}$$

and $h^{\mu\nu}(\vec{x},t)$ is given by eq.(5.11) and eqs.(5.16)-(5.36). From these expressions we see that non-relativistic particles experience a force due to the mass term $h^{00}_a(\vec{x},t)$, the terms $h^{0i}_b(\vec{x},t)$, $\dot{h}^{0i}_b(\vec{x},t)$ and the term $\ddot{h}^{00}_d(\vec{x},t)$. Thus, eq.(6.26) can be written in the following way:

$$\begin{aligned}\mathcal{F}(\vec{x},t)\hat{x}^i = \frac{d^2x^i}{dt^2} \;&=\; 2G\left[\frac{8\pi}{3}\hat{x}^i\left(\frac{d\mathcal{G}_1}{dt} + r\frac{d^2\mathcal{G}_1}{dt^2}\right)\right.\\ &\left. -\frac{dh^{00}_a}{dr}\,\hat{x}^i - \frac{1}{2}\frac{d\ddot{h}^{00}_d}{dr}\,\hat{x}^i\right].\end{aligned} \tag{6.28}$$

From the discussion in the last section we know that the dominant term in the above expression is

$$\frac{dh^{00}_a}{dr}\,\hat{x}^i$$

but let us keep all the terms in eq.(6.28) just to be sure we do not miss any additional information in its behaviour. The exact expressions from these contributions are not difficult to obtain, however, these expressions are exceedingly lengthy, for this reason we have not included them here (the technical details of this computation and the corresponding expressions for the gravitational force exerted by the string on non-relativistic particles can be found in appendix A.) Here we discuss their behaviour in a schematic way.

As we can see from figure (6.11), if we fix the position and plot the gravitational force as time evolves, we find that non-relativistic particles will experience small attractive and repulsive forces. Therefore we must take in such a case a time average of the force exerted on the particles to see how the particles behave overall:

$$\mathcal{F}_{avg}(\vec{x})\hat{x}^i = \overline{\mathcal{F}(\vec{x})}\hat{x}^i = \frac{\hat{x}^i}{J}\int_0^J dt_{ret}\,\mathcal{F}(\vec{x},t_{ret}) \tag{6.29}$$

where J is the time elapsed in a few periods of oscillation. From figure (6.11) we might want to take a value for J equal to $J = 500$. We can see now in figure (6.12) that

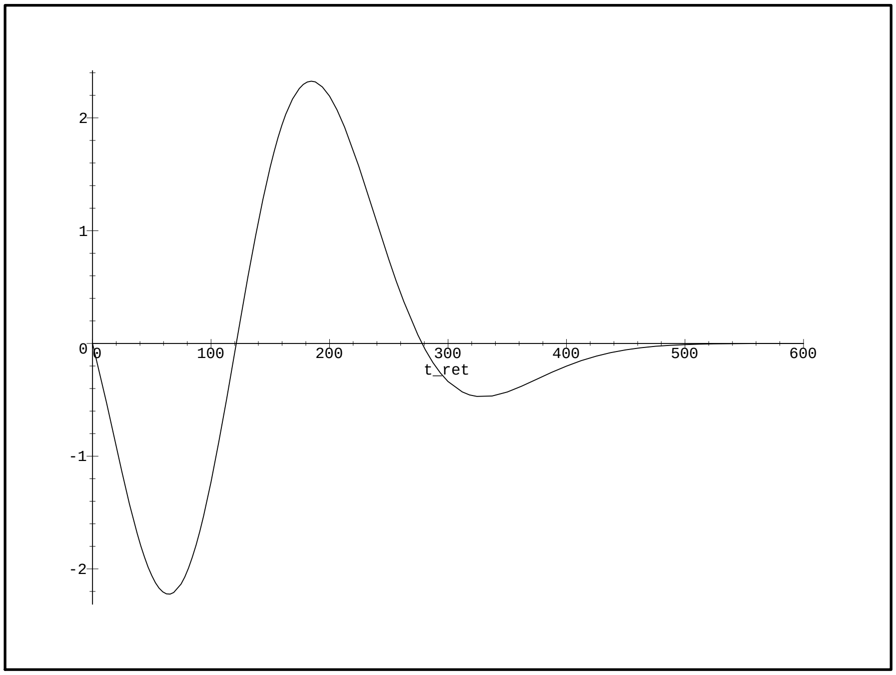

Figure 6.11: The gravitational force exerted by a quantum string on non-relativistic particles on the $\hat{x}^i$ direction. (r fixed and equal to 100 and $\alpha = 10000$. In units of $G \times 10^{-9}$.)

non-relativistic particles experience a small attractive force. The resultant force being:

$$\begin{aligned} F_{res}(\vec{x}) &= \mathcal{F}_{avg}(\vec{x})\hat{x}^1 + \mathcal{F}_{avg}(\vec{x})\hat{x}^2 + \mathcal{F}_{avg}(\vec{x})\hat{x}^3 \\ &= -\sqrt{3}\left|\overline{\mathcal{F}(\vec{x})}\right| \hat{r} \end{aligned} \tag{6.30}$$

Its plot is given in fig.(6.13).

6.4 Remarks

We have seen that considering the quantum nature of massless bosonic strings provides us with several new and interesting features in respect to cosmic strings. Some of these features are:

1. Our weak-field approximation for the metric derived from the low energy excitation of a quantum bosonic string holds in the region under study namely the far field limit ($r \gg R$). The metric computed in chapter 5 approaches that of flat Minkowski space-time as we go to distances far from the source. The divergent logarithmic behaviour of cosmic strings [97] is not present here.

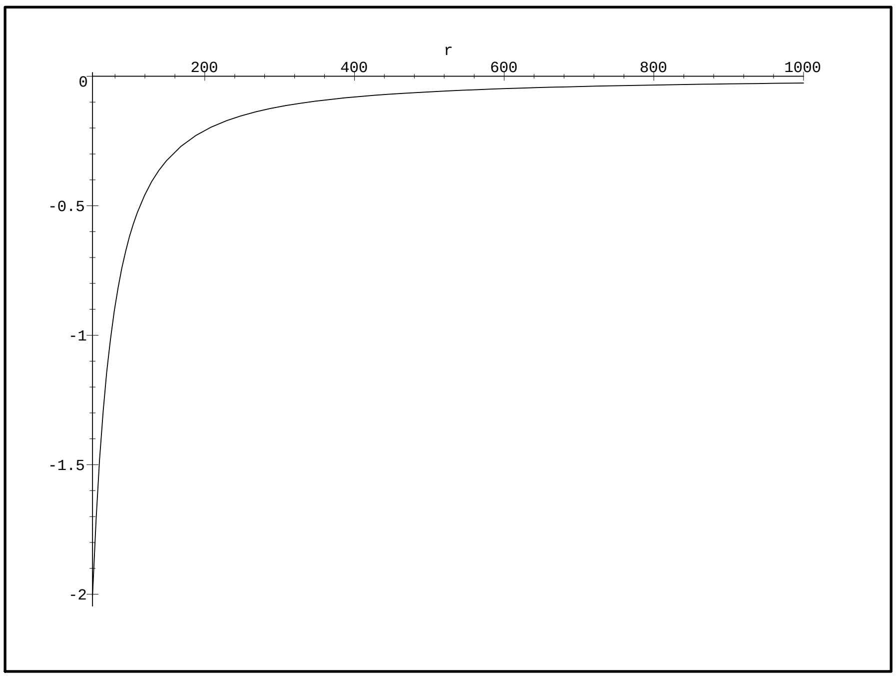

Figure 6.12: The averaged gravitational force exerted by a quantum string on non-relativistic particles on the $\hat{x}^i$ direction. ($J = 1000$ and $\alpha = 10000$. In units of $G \times 10^{-14}$.)

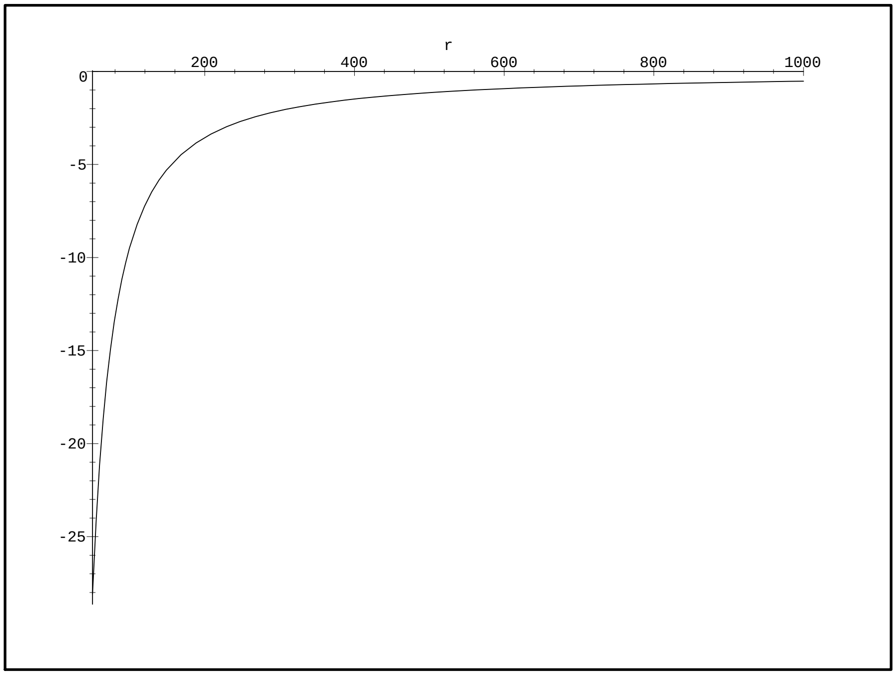

Figure 6.13: The averaged resultant gravitational force exerted by a quantum string on non-relativistic particles. Here $J = 500$, $\alpha = 10000$ (In units of $G \times 10^{-13}$)

2. The metric presented in chapter 5 and studied in this chapter, does not present an equivalent 'deficit angle' as the metric for cosmic string does (at least not in any obvious way). Since quantum strings are fundamentally very different from cosmic strings, we should not be expecting this to be the case.

3. The space-time geometry generated by a quantum bosonic string presents small anisotropies introduced by the higher order terms in the multipole expansion of our metric namely: h_b^{0i}, h_c^{ii} $\dot{h}_b^{0i}$, $\dot{h}_c^{ii}$ and $\ddot{h}_d^{ii}$. These terms may be of interest from a Cosmological point of view.

We found that the behaviour of the gravitational force as described here is not too different from some of the behaviour that can be found for cosmic strings (see for example [104]), which considered classical non-static strings. In [104] it was found that the gravitational field of a non-static string is the one given by its total mass. That would have been the case here as well if we neglected all the terms in eq.(6.28) which are subdominant in our slow motion approximation which is what is done in [104], however we have chosen to keep all the relevant terms in eq.(6.28). We can see this more clearly by looking at eqs.(5.12)-

(5.15). The subdominant terms are given by eqs.(5.13)-(5.15) and to compute the resultant gravitational force we have performed an average over the time; thus, integrating eq.(5.12):

$$h_a^{\mu\nu}(\vec{x},t) = \frac{4G}{r}\int d^3\vec{x}'\langle\hat{T}^{\mu\nu}(\vec{x}',t_{ret})\rangle, \tag{6.31}$$

over t_{ret} we find that the RHS of this expression is just $constant/r$ which can be interpreted as usual [94, 95] as M/R, M being the total mass of the source. Hence to leading order, the gravitational force associated with this term is the one given by the total mass of the source.

To close the analysis of these results, let us recall that the linearised Einstein's equations are based upon the assumption that we possess sufficient knowledge of the behaviour of our source. Furthermore, we have made also the assumption that most of the matter is concentrated within a certain region of radius R and we have neglected contribution from matter outside this region. These assumptions, may or may not be true. Further study of the behaviour of quantum strings in the space-time geometry presented in this chapter and the previous one will throw more light on the gravitational properties of quantum strings. However, the intention of this work has been mainly to outline the differences between the behaviour of classical strings and their quantum counterpart and to present a starting point for future developments in this direction.

CHAPTER 7

Quantum bosonic strings in shock-wave space-times

In this chapter we will present some results on quantum bosonic strings in a shock-wave background. Strings in this type of space-time configuration have been studied in detail by a number of authors [10, 86], [107]-[110]. Shock-wave space-times are important to consider because we may interpret this geometry as the one produced by an ultra-relativistic particle; therefore, they present an opportunity to study the scattering between strings and ultra-relativistic particles close to the Planck energy scale, where the gravitational interaction is the one that dominates the scattering process. These geometries are also important because they can give us a description of the geometry upon which a string moves when it is in the presence of other strings [10] (in the previous chapters we considered only isolated strings.)

In this chapter we will try to show in a schematic way that at the quantum level the string energy-momentum tensor in a shock-wave space-time configuration leads to extra terms which result from the excitation of the oscillation modes of the string when the string collides with the shock-wave. We will also show some basic differences between the results computed in [10] and ours, which maintain the string nature of the energy-momentum tensor. (In [10] the authors integrated the string energy-momentum tensor over a volume completely enclosing the string; thus, loosing its stringy features.) As we will see, these extra terms are important to consider since their contributions to the string energy-momentum tensor seem to come from all the possible interactions of the oscillation modes of the string which are being excited by the shock-wave.

7.1 The Aichelburg-Sexl geometry

Let us very briefly summarise some of the results of the Aichelburg-Sexl geometry. In [108] the Aichelburg-Sexl geometry was interpreted as that of a shock-wave produced by

a neutral and spinless ultra-relativistic particle. This geometry has been generalised by a number of authors so as to include ultra-relativistic particles with charge and spin. (See for example [109, 110]).

The Aichelburg-Sexl metric can be made to take the following form:

$$dS^2 = dUdV - (dX^i)^2 + f(\rho)\delta(U)dU^2. \tag{7.1}$$

(The derivation of this metric can be found in [111] and an alternative computation is presented also in [108].) Here U and V are given by [10]:

$$U = X^0 - X^1$$

and

$$V = X^0 + X^1.$$

In addition we have that $\rho = |X^i|$, $2 \leq i \leq D-2$, and $f(\rho)$ satisfies the following expression:

$$\nabla_\perp^2 f = 16\pi G\rho(X^i)$$

where $\nabla_\perp$ refers to the transverse part of ∇. As we can see, the space-time is everywhere flat except at the location of the shock-wave $U = 0$. Some of the main features of this type of metric are:

1. The Aichelburg-Sexl metric is basically that of a boosted black-hole metric where we demand that the momentum of the ultra-relativistic particle remains constant as it approaches the speed of light. That is, we demand that its mass $m \rightarrow 0$ as its velocity $v \rightarrow c$.

2. Geodesics in this metric have a discontinuity at the intersection with the shock-wave.

3. There exists a 'time shift' owing to the discontinuity mentioned above. Clocks slow down as the shock-wave passes through them.

4. There exists a 'spatial refraction' as well. Objects are pushed towards the trajectory of the ultra-relativistic particle. Objects are moved also towards the general direction of the shock-wave.

7.2 The string non-linear transformations

In this section we will present the main results found in the literature on bosonic strings in shock-wave space-time configurations [86], [107]-[110]. The equations of motion for the strings are highly non-linear owing to the curvature of the space-time. In the light-cone gauge $U = 2\alpha' p^u \tau$ one finds that the string equations of motion are the same as those for a flat space-time when $\tau > 0$ and $\tau < 0$, $\tau = 0$ being the moment when the string interacts with the shock-wave. Thus, the string coordinates are given by:

$$X^\mu(\sigma,\tau) = q^\mu_< + 2\alpha' p^\mu_< \tau + i\sqrt{\alpha'} \sum_{n\neq 0} \frac{1}{n}[\alpha^\mu_{n<}\ e^{-in(\tau-\sigma)} + \tilde{\alpha}^\mu_{n<}\ e^{-in(\tau+\sigma)}] \tag{7.2}$$

and by:

$$X^\mu(\sigma,\tau) = q^\mu_> + 2\alpha' p^\mu_> \tau + i\sqrt{\alpha'} \sum_{n\neq 0} \frac{1}{n}[\alpha^\mu_{n>}\ e^{-in(\tau-\sigma)} + \tilde{\alpha}^\mu_{n>}\ e^{-in(\tau+\sigma)}]. \tag{7.3}$$

The transformations that take us from the $<$ region to the $>$ are given by [10], [108]:

$$q^i_> - q^i_< = 0, \tag{7.4}$$

$$p^i_> - p^i_< = i\frac{p^u}{4\pi}\int_0^{2\pi} d\sigma \int d^{D-2}pp^i\varphi(\vec{p}) : e^{i\vec{p}\cdot C(\sigma)} :, \tag{7.5}$$

$$\alpha^i_{n>} - \alpha^i_{n<} = i\frac{\sqrt{\alpha'}p^u}{4\pi}\int_0^{2\pi} d\sigma \int d^{D-2}pp^i\varphi(\vec{p}) : e^{i\vec{p}\cdot C(\sigma)} : e^{in\sigma}, \tag{7.6}$$

$$\tilde{\alpha}^i_{n>} - \tilde{\alpha}^i_{n<} = i\frac{\sqrt{\alpha'}p^u}{4\pi}\int_0^{2\pi} d\sigma \int d^{D-2}pp^i\varphi(\vec{p}) : e^{i\vec{p}\cdot C(\sigma)} : e^{-in\sigma}. \tag{7.7}$$

Here, $C(\sigma)$ is given by:

$$C^i(\sigma) = X^i_<(\sigma,\tau=0) = X^i_>(\sigma,\tau=0) = q^i + i\sqrt{\alpha'}\sum_{n\neq 0}\frac{e^{in\sigma}}{n}[\alpha^i_{n<} - \tilde{\alpha}^i_{n<}],$$

and $\varphi(\vec{p})$ is related to the matter-density of the source by means of a Fourier transformation:

$$\varphi(\vec{p}) = -\frac{16\pi G}{p^2}\int \frac{d^{D-2}X}{(2\pi)^{D-2}}\rho(X)e^{-i\vec{p}\cdot X}.$$

7.3 The string energy-momentum tensor in a shock-wave space-time

In this section we want to outline some of the main differences between the results we computed in chapter 3 for strings in a Minkowski space-time metric and those for strings in a shock-wave space-time metric. We will also see that keeping the stringy nature of the energy-momentum tensor will lead to very different results from those found in [10],

where the authors integrated the string energy-momentum tensor over a volume completely enclosing the string.

Consider the string energy-momentum tensor:

$$T^{\mu\nu}(x) = \frac{1}{2\pi\alpha'} \int d\sigma d\tau \; (\dot{X}^{\mu}\dot{X}^{\nu} - X'^{\mu}X'^{\nu})\delta(x - X(\sigma,\tau)). \tag{7.8}$$

We are now interested in computing the quantum expectation value of $T^{\mu\nu}(x)$ once the string has collided with the shock-wave with respect to the ground state. That is, we want to compute the expectation value $\langle 0_{<}|\hat{T}^{\mu\nu}_{>}(x)|0_{<}\rangle$.

7.3.1 Keeping the string nature of $\langle \hat{T}^{\mu\nu}(x)\rangle$

To start, let us examine the delta function appearing in eq.(7.8). We can write it as:

$$\delta(x - X_{>}(\sigma,\tau)) = \frac{1}{(2\pi)^D} \int d^D\lambda e^{i\lambda\cdot(x - X_{>}(\sigma,\tau))} \tag{7.9}$$

just as we did in chapter 3. We can see that expression (7.9) is not a trivial one: we can write $X_{>}(\sigma,\tau)$ as:

$$X_{>}(\sigma,\tau) = X_{>cm} + X_{>-} + X_{>+},$$

where $-$ and $+$ refer to the creation and annihilation parts of $X_{>}(\sigma,\tau)$ and cm to the centre of mass coordinates: $q_{>} + 2\alpha' p_{>}\tau$. Expressing all the $>$ operators in terms of the $<$, which are those we know how to work with, we find that in addition to the usual terms we presented in chapter 3 we also have terms which represent the interaction of the centre of mass coordinates and the oscillation modes of the string with the shock-wave of an ultra-relativistic particle.

In chapter 3 we found that normal ordering eq.(7.9), only the centre of mass coordinates of this operator contributes to the energy-momentum expectation value. Here, however, we can see that is not the case. The additional terms representing the interaction of the string with the shock-wave do not vanish and will give extra contributions to the expectation value. Furthermore, the centre of mass coordinates will also present extra terms from the string interaction with the shock-wave. Notice that none of these additional terms would be present if we integrated over a spatial volume enclosing the string. Therefore, these 'additional' terms are effects arising from the extended nature of strings.

After some work we find that the terms in brackets in eq.(7.8) can be written as:

$$\dot{X}^{\mu}_{>}(\sigma,\tau)\dot{X}^{\nu}_{>}(\sigma,\tau) - X'^{\mu}_{>}(\sigma,\tau)X'^{\nu}_{>}(\sigma,\tau) = 4\alpha'^2 p^{\mu}_{>}p^{\nu}_{>} + 2(\alpha')^{3/2}p^{\mu}_{>} \times$$

$$\sum_{m\neq 0} e^{-im\tau}[\alpha^{\nu}_{m>}e^{im\sigma} + \tilde{\alpha}^{\nu}_{m>}e^{-im\sigma}] + 2(\alpha')^{3/2}\sum_{n\neq 0} e^{-in\tau}[\alpha^{\mu}_{n>}e^{in\sigma} + \tilde{\alpha}^{\mu}_{n>}e^{-in\sigma}]p^{\nu}_{>} +$$
$$\alpha'\sum_{n\neq 0}\sum_{m\neq 0} e^{-i(n+m)\tau}[2\alpha^{\mu}_{n>}\tilde{\alpha}^{\nu}_{m>}e^{i(n-m)\sigma} + 2\tilde{\alpha}^{\mu}_{n>}\alpha^{\nu}_{m>}e^{-i(n-m)\sigma}] \quad (7.10)$$

Of course, in order to take the expectation value we need to normal order the above expression.

In order to obtain a better insight into the situation we are working with, let us just look at the first term of eq.(7.10). If we express this term in terms of the $<$ operators, we obtain (setting $\alpha' = 1/2$):

$$\begin{aligned} p^{\mu}_{>}p^{\nu}_{>} \quad = \quad & p^{\mu}_{<}p^{\nu}_{<} + \left[-\frac{p^{u}}{16\pi^2}\int_0^{2\pi} d\sigma \int d^{D-2}pp^i\varphi(\vec{p}) : e^{i\vec{p}\cdot C(\sigma)} : \delta_{\mu,i}\right] \times \\ & \left[p^{u}\int_0^{2\pi} d\sigma' \int d^{D-2}p'p^j\varphi(\vec{p}') : e^{i\vec{p}'\cdot C(\sigma')} : \delta_{\nu,j}\right] + \\ & \left[i\frac{p^{u}}{4\pi}\int_0^{2\pi} d\sigma \int d^{D-2}pp^i\varphi(\vec{p}) : e^{i\vec{p}\cdot C(\sigma)} : \delta_{\mu,i}\right] p^{\nu}_{<} + \\ & i\frac{p^{\mu}_{<}p^{u}}{4\pi}\int_0^{2\pi} d\sigma' \int d^{D-2}p'p^j\varphi(\vec{p}') : e^{i\vec{p}'\cdot C(\sigma')} : \delta_{\nu,j}. \end{aligned} \quad (7.11)$$

It is to be noticed that upon integration of this expression (with the delta function factor included as well) over a spatial volume we would obtain the same results presented in [10].

7.4 A brief analysis of the string energy-momentum tensor expectation value in a shock-wave space-time configuration

It has not been the intention in this thesis to compute or present explicit results for the expectation value of the string energy-momentum tensor in shock-wave space-times since the highly non-linear transformations make such a task extremely difficult to achieve. Instead, from here on, we will proceed very schematically. We will try to show what additional terms emerge as a consequence of keeping the extended nature of the string and examine their relevance.

From eq.(7.11) we see that the first term of this expression is the term we worked with in the previous chapters of the thesis. At this point, we may be tempted to think that we will obtain our results obtained in chapter 3 and that the other terms in eq.(7.11) are just additional terms arising from the fact that we are now in a non-trivial background. However, this is not entirely true. Certainly, the extra terms in eq.(7.11) are additional to the expectation value of the energy-momentum tensor we computed in chapter 3 but even

if these terms were not present and we concentrated only on the first one, we would still obtain different results here. Such a term would give a completely different contribution from the one we found previously, the reason being that the delta function present in the energy-momentum tensor turns out to be a very complex quantum operator.

Let us consider, just as an example, the first term in expression (7.11). Its expectation value is given by:

$$\langle p^{\mu}_{<} p^{\nu}_{<} \rangle = \langle 0_{<} | e^{-i\lambda \cdot X_{-}} p^{\mu}_{<} p^{\nu}_{<} e^{i\lambda \cdot \vec{x}} e^{-i\lambda \cdot X_{cm}} e^{-i\lambda \cdot X_{+}} | 0_{<} \rangle. \tag{7.12}$$

Here $e^{-i\lambda \cdot X_{cm}}$ is given by:

$$e^{-i\lambda \cdot X_{cm}} = e^{-i\lambda \cdot (q^{\beta}_{<} + p^0 \tau \delta_{\beta,0} + p^1 \tau \delta_{\beta,1} + p^i_{<} \tau \delta_{\beta,i} + i \frac{\tau p^u}{8\pi} \int_0^{2\pi} d\sigma \int d^{D-2}p p^i \varphi(\vec{p}) : e^{i\vec{p} \cdot C(\sigma)} : \delta_{\beta,i})}, \tag{7.13}$$

where the repeated indexes do not represent implicit sums. The $e^{-i\lambda \cdot X_{\pm}}$ are given by:

$$\begin{aligned} e^{-i\lambda \cdot X_{\pm}} &= e^{-i\lambda \cdot (X_{R\pm} + X_{L\pm})} \\ & e^{\pm \frac{\lambda}{\sqrt{2}} \cdot (\sum_{s=0} \frac{e^{\mp i s \tau}}{s} (\alpha^0_{\pm s <} \delta_{\beta,0} + \alpha^1_{\pm s <} \delta_{\beta,1} + [\alpha^i_{\pm s <} + i \frac{p^u}{4\sqrt{2}\pi} \int_0^{2\pi} d\sigma \int d^{D-2}p p^i \varphi(\vec{p}) : e^{i\vec{p} \cdot C(\sigma)} : e^{\pm i s \sigma}]) \delta_{\beta,i} \, e^{i n \sigma}) + \ldots}, \end{aligned} \tag{7.14}$$

and $(\ldots)$ is a similar expression involving the left-moving operators of the string. From this expression we can clearly see that the expectation value of a quantum bosonic string in a curved space-time background, in particular in a shock-wave space-time, will contain very complex terms arising from the interaction of the string with the shock-wave. And remembering from chapter 3 that the delta function in the string energy-momentum tensor, when considered a quantum operator, is basically of the same form of a vertex operator, we see that it seems we have got here an expansion in terms of vertex operators. When the string collides with the shock-wave all the oscillation modes of the string become excited. In other words, when we keep the stringy features of the string energy-momentum value, $\langle \hat{T}^{\mu\nu}(x) \rangle$ takes into account all possible interactions between the string and the shock-wave plus all the interactions emerging from the oscillation modes excited by the shock-wave:

$$\begin{aligned} \delta(x - X(\sigma,\tau)) &= \frac{1}{(2\pi)^D} \int d^D\lambda e^{i\lambda \cdot \vec{x}} e^{-i\lambda \cdot (X^{\mu} + V_{\beta})} \\ &= \frac{1}{(2\pi)^D} \int d^D\lambda e^{i\lambda \cdot \vec{x}} e^{-i\lambda \cdot X^{\mu}} \left(1 - i\lambda \cdot V_{\beta} + \frac{(i\lambda \cdot V_{\beta})^2}{2!} - \frac{(i\lambda \cdot V_{\beta})^3}{3!} + \ldots \right) \times \\ & \quad e^{-\lambda^2 [X, V_{\beta}]}. \end{aligned} \tag{7.15}$$

Here V_β is the nonlinear transformation between the $<$ and $>$ operators of the string. This transformation is nothing other than the "... transverse part of a vertex operator $e^{i\vec{p}\cdot X(\sigma)}$ integrated over p and over the world-sheet at $\tau = 0$ " [10]. The importance of not losing the stringy nature of $\langle \hat{T}^{\mu\nu}(x)\rangle$ now seems to be clear. With this expression we can now see that the expectation value given by eq.(7.12) will take the following form:

$$\begin{aligned}
\langle p^\mu_< p^\nu_< \rangle &= \frac{1}{(2\pi)^D}\int d^D\lambda \langle 0_< | e^{-i\lambda\cdot(X^\mu_- + V_{\beta-})} p^\mu_< p^\nu_< e^{i\lambda\cdot\vec{x}} e^{-i\lambda\cdot(X^\mu_+ + V_{\beta+})}|0_<\rangle \\
&= \frac{1}{(2\pi)^D}\int d^D\lambda \langle 0_< | e^{i\lambda\cdot\vec{x}} e^{-i\lambda\cdot X^\mu_-}\left(1 - i\lambda\cdot V_{\beta-} + \frac{(i\lambda\cdot V_{\beta-})^2}{2!} - \frac{(i\lambda\cdot V_{\beta-})^3}{3!} + \ldots\right)\times \\
&\quad e^{-\lambda^2[X_-,V_{\beta-}]}\, p^\mu_< p^\nu_<\, e^{-i\lambda\cdot X^\mu_{cm}} e^{-i\lambda\cdot X^\mu_+}\times \\
&\quad \left(1 - i\lambda\cdot V_{\beta cm} + \frac{(i\lambda\cdot V_{\beta cm})^2}{2!} - \frac{(i\lambda\cdot V_{\beta cm})^3}{3!} + \ldots\right)\times \\
&\quad \left(1 - i\lambda\cdot V_{\beta+} + \frac{(i\lambda\cdot V_{\beta+})^2}{2!} - \frac{(i\lambda\cdot V_{\beta+})^3}{3!} + \ldots\right) e^{-\lambda^2[X_+,V_{\beta+}]}|0_<\rangle.
\end{aligned} \tag{7.16}$$

It is important to notice that the result we have just presented above would not have arisen if we had integrated over a spatial volume totally enclosing the string.

7.5 Remarks

In this chapter we have tried to present very briefly some of the main differences between the computation for the string energy-momentum tensor presented in [10] and ours. Whilst in [10] the string energy-momentum tensor was integrated over a spatial volume and therefore we no longer had all the string features of $\langle \hat{T}^{\mu\nu}(x)\rangle$, here, we have kept all the string features of it. In doing this we find that the expectation value presents contributions from all the interactions take place when the string collides with a shock-wave. The 'price' we have to pay for this is that explicit calculations are exceedingly difficult to perform.

As a final note, it has to be remembered that shock-wave space-times are not candidates for the string vacua. The shock-wave metric does not satisfy the conditions required by conformal invariance [10].

CHAPTER 8

Conclusions

In this work we have attempted to provide the preliminaries for a more sophisticated study of the gravitational properties of fundamental strings. We have shown in this thesis that the behaviour of the energy-momentum tensor when treated as a quantum operator presents features which are different from its classical counterpart.

In this thesis, we wanted to outline the quantum results of a bosonic string in the context of the gravitational properties they present. We asked the question: Is the behaviour of fundamental strings at the quantum level similar to that of a classical string (e.g. a cosmic string)? As we have seen throughout the chapters of this thesis, the answer is no, there are a number of differences that certainly deserve further study since their interpretation is still far from obvious. Let us recall what we hope we have learnt from the chapters of this thesis.

The main differences obtained with respect to the classical theory of strings are:

1. As shown in chapter 3, the energy-momentum tensor loses its locality. This happens because quantum fluctuations smear the string position. This is a purely quantum mechanics effect and it is in some sense a reflection of Heisenberg's principle of uncertainty.

2. The energy-momentum tensor at the quantum level behaves roughly like a string vertex operator. In this thesis we have concentrated mainly on the string massless states; that is, photons, gravitons and dilatons, although we have written down solutions for massive states.

3. The string energy-density for massless string states decays as $1/r$ when $r \to \infty$ and t fixed. We showed in chapters 3 and 4 that the total energy formally diverges in this regime only if we work in a temporal box which is very small compared to the

radius of the volume of space, an acausal situation. On the other hand in the causal limit $r < t$, the energy remained finite.

After analysing in detail the string energy-momentum tensor, we proceeded to study the gravitational properties of quantum bosonic strings in a first order approximation to Einstein's field equations. Because of the difficulties presented by the very complex structure of the quantum string energy-momentum tensor, we studied a multipole expansion of the weak-field metric $h^{\mu\nu}(x)$ in the far field limit, in a way analogous to that which can be performed in the electro-magnetic theory. We showed in chapter 6 that these properties are different in form and in content from those found for classical strings (cosmic strings):

1. We found that in the quantum counterpart there exists no obvious relation between potential divergences that may have emerged there and the position of the string since the string position has been smeared out by quantum fluctuations. This is a purely quantum effect.

2. We also found that because fundamental strings are of a quantum nature and move at relativistic speeds, static solutions like those found by Vilenkin [97] are no longer relevant. We found that the metric obtained for a quantum bosonic string does not resemble in any obvious way the metric for a static classical string (cosmic string) found by Vilenkin. We showed that $h^{\mu\nu}(x)$ falls away at large distances a result consistent with the weak field assumptions made.

3. We studied the gravitational radiation produced by a quantum string and found that the string radiates in the form of quadrupole radiation and that there is no dipole radiation. This is in total agreement with standard results in general relativity.

4. We compute the gravitational force exerted by a massless bosonic string on non-relativistic particles. We found that when the force is integrated over periods of time the force is that given by the total mass of the source. This results is similar to the one found by [104].

5. Our results represent a semi-classical approximation in that the energy-momentum source in Einstein's equations is of a quantum nature, entering via its expectation value. The expectation value was taken with respect to the low excitations of a quantum bosonic string; that is, the states in the expectation value represent massless string states such as gravitons, photons and dilatons.

From the points mentioned above, it is clear that much more work is needed in order to assess the relevance of quantum strings in Cosmology. As we have said elsewhere, this work has been developed as an approach to understanding the gravitational properties of quantum strings.

The study of the gravitational properties of quantum bosonic strings in non-trivial space-times has already been contemplated. In particular we are looking at bosonic strings in a shock-wave background (for which some ideas were presented in the previous chapter). In this case the situation is expected to be much more complicated since there are non-trivial transformations between the string oscillation modes before the string collides with the shock-wave and after the string has collided. We expect to report on that case also at some time in the future.

BIBLIOGRAPHY

[1] Freedman, von Nieuwenhuizen and Ferrara, *Phys. Rev.* **D13**, 3214 (1976).

[2] S. Ferrara, Z. D. Freedman, P. van Nieuwenhuizen, P. Breitenlohner, F. Gliozzi and J. Scherk, *Phys. Rev.* **D15**, 1013 (1977).

[3] S. Ferrara, J. Wess and B. Zumino, *Phys. Lett.* **51 B**, 239 (1974).

[4] S. Ferrara, F. Gliozzi, J. Scherk and P. van Nieuwenhuizen, *Nucl. Phys*, **B117**, 333 (1976).

[5] S. Desser and B. Zumino, *Phys. Rev. Lett.* **38**, 1433 (1977).

[6] S. Desser and B. Zumino, *Phys. Lett.* **62 B**, 335 (1976).

[7] S. Weinberg, *Phys. Rev. Lett.* **19**, 1264 (1967).

[8] A. Salam in 'Elementary Particle Theory', Proceedings of the 8th Nobel Symposium, Stockholm 1968.

[9] H. Georgi and S. L. Glashow, *Phys. Rev. Lett.* **32**, 438 (1974).

[10] H. J. de Vega, *Int. J. Mod. Phys.* **A7**, 3043 (1992).

[11] H. J. de Vega in Proceedings of the Erice School: "String Quantum Gravity and Physics at the Planck Energy Scale", 21-28 June 1992, Edited by N. Sánchez, World Scientific, 1993. Edited by N. Sánchez and A. Zichichi, Kluwer Acad. Publ.

[12] A. Vilenkin in Proceedings of the Erice School: "String Gravity and Physics at the Planck Energy Scale", 8-19 September 1995, Edited by N. Sánchez, World Scientific, 1996. Edited by N. Sánchez and A. Zichichi, Kluwer Acad. Publ.

[13] M. Davis, *Preprint astro-ph 9610149v2*, (1996).

[14] L. Pietronero *et al. Preprint astro-ph 9611197*, (1996).

[15] A. Dressler, *Astrophys. J.* **329**, 519 (1988).

[16] A. Dressler and S. M. Faber, *Astrophys. J.* **354**, L45 (1990).

[17] A. Dressler and S. M. Faber, *Astrophys. J.* **354**, 13 (1990).

[18] J. A. Willick, *Astrophys. J.* **351**, L5 (1990).

[19] B. I. Hnatyk, V. N. Lukash and B. S. Novosyadlyj, *Astr. Astrophys.* **300**, 1 (1995).

[20] J. M. Bardeen, J. R. Bond and G. Efstathion, *Astrophys. J.* **321**, 28 (1987).

[21] D. Burstein, A. Dressler and S. M. Faber, *Astrophys. J.* **354**, 18 (1990).

[22] Yu. V. Baryshev *et al. Preprint astro-ph 9503074*, (1995).

[23] E. Martinez-Gonzalez and J.L. Sanz, Astr. Astrophys. **300**, 346 MS, (1995).

[24] R. Marteens, G. F. R. Ellis and W. R. Stoeger, *Preprint astro-ph 9510126*, (1995).

[25] R. Durrer and J. Laukenmann, *Class. Quant. Grav.* **13**, 1069 (1996).

[26] W. H. Kinney and K. T. Mahanthappa, *Phys. Rev.* **D 53**, 5455 (1996).

[27] R. F. Langbein, K. Langfeld, H. Reinhardt and L. von Smekal, *Mod. Phys. Lett.* **A11**, 631 (1996).

[28] E. W. Kolb and M. S. Turner, *The Early Universe*, Addison Wesley (1994).

[29] S. F. Smoot *et al.*, *Astrophys. J.* **396**, L1 (1992).

[30] E. L. Wright *et al.*, *Astrophys. J.* **396**, L13 (1992).

[31] J. B. Hartle and S. W. Hawking, *Phys. Rev.* **D 28**, 2960 (1983).

[32] A. Vilenkin, *Phys. Rev.* **D 30**, 509 (1984).

[33] A. D. Linde, *Lett. Nuovo Cim.* **39**, 401 (1984).

[34] H. Luckock, I. Moss and D. Toms, *Nucl. Phys.* **B 297**, 748 (1988).

[35] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-time*, Cambridge University Press (1984).

[36] R. H. Dicke and P. J. E. Peebles in General Relativity: An Einstein Centenary Survey. Edited by S. W. Hawking and W. Israel, Cambridge University Press (1979).

[37] S. Lansdale and I. Moss, *Nucl. Phys.* **B 298**, 693 (1988).

[38] S. Carlip, *Phys. Rev.* **D 46**, 4387 (1992).

[39] S. Carlip, *Class. Quant. Grav.* **10**, 1057 (1993).

[40] A. O. Barbinsky, A. Yu. Kamenshchik, *Class. Quant. Grav.* **7**, L181 (1990).

[41] L. P. Grishchuk and L. V. Rozhansky, *Phys. Lett.* **B 234**, 9 (1990).

[42] A. O. Barbinsky, A. Yu. Kamenshchik and I. P. Karmazin, *Anns. Phys.* **219**, 201 (1992).

[43] H. Nariai and K. Tomita, *Prog. Theor. Phys.* **46**, 776 (1971).

[44] L. Parker and S. A. Fulling, *Phys. Rev.* **D7**, 2357 (1973).

[45] D. Atkatz, and H. Pagels, *Phys. Rev.* **D 25**, 2065 (1982).

[46] E. P. Tryon, *Nature*, **246**, 396 (1973).

[47] R. Brout, *et al.*, *Nucl. Phys.* **B170**, 228 (1980).

[48] R. Brout, F. Englert and E. Gunzig, *Ann. Phys.* **115**, 78 (1978).

[49] P. Candelas, G. T. Horowitz, A. Strominger and E. Witten, *Nucl. Phys.* **B 258**, 46 (1985).

[50] D. Bailin, A. Love and S. Thomas, *Phys. Lett.* **B 176**, 81 (1986).

[51] D. Bailin, A. Love and S. Thomas, *Phys. Lett.* **B 178**, 15 (1986).

[52] D. Bailin, A. Love and S. Thomas, *Nucl. Phys.* **B 273**, 537 (1986).

[53] D. Bailin, and A. Love, *Phys. Lett.* **B 157**, 375 (1985).

[54] A. S. Majuundar, T. R. Seshadri and S. K. Sethi, *Phys. Lett.* **B 312**, 67 (1993).

[55] D. Bailin, and A. Love, *Phys. Lett.* **B 207**, 151 (1988).

[56] D.Bailin, D. C. Dunbar and A. Love, *Int. J. Mod. Phys.* **A5**, 939 (1990).

[57] A. S. Majuundar, A. Mukherjee and R. P. Saxena, *Mod. Phys. Lett.* **A7**, 3647 (1992).

[58] D. Bailin, G. V. Kraniotis and A. Love, *Preprint hep-th 9705244v2*, (1997).

[59] J. H. Schwarz in "Superstrings and Supergravity" Proceedings of the Twenty-Eighth Scotish Universities Summer School in Physics 1985. Edited by A. T. Davies and D. S. Sutherland.

[60] V. Alessandrini, D. Amati, M. LeBellac, and D. I. Olive, *Phys. Rep.* **1C**, 170 (1971).

[61] S. Mandelstam, *Phys. Rep.* **13C**, 259 (1974).

[62] G. Veneziano, *Phys. Rep.* **9C**, 199 (1974).

[63] J. H. Schwarz, *Phys. Rep.* **8C**, 269 (1973).

[64] A. M. Polyakov, *Phys. Lett.* **B 103**, 207, 211 (1981).

[65] P. Ramond, *Phys. Rev.* **D 3**, 2415 (1971).

[66] A. Neveu and J. H. Schwarz, *Nucl. Phys.* **B 31**, 816 (1971).

[67] A. Neveu and J. H. Schwarz, *Phys. Rev.* **D 4**, 1109 (1971).

[68] K. Kikkawa, B. Sakita, and M. A. Virasoro, *Phys. Rev.* **184**, 1701 (1969).

[69] M. Kaku and L. P. Yu, *Phys. Lett.* **B 33**, 166 (1970).

[70] M. Kaku and L. P. Yu, *Phys. Rev.* **D 3**, 2992, 3007, 3020 (1971).

[71] M. Kaku and J. Scherk, *Phys. Rev.* **D 3**, 430 (1971).

[72] M. Kaku and J. Scherk, *Phys. Rev.* **D 3**, 2000 (1971).

[73] C. Lovelace, *Phys. Lett.* **B 34**, 500 (1971).

[74] T. Goto, *Phys. Theor. Phys.* **46**, 1560 (1971).

[75] J. Scherk, *Rev. Mod. Phys.* **47**, 123 (1975).

[76] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory*, Cambridge University Press, (1985).

[77] M. Kaku, *Introduction to Superstring Theory*, Spring-Verlag, (1988).

[78] J. M. López R., M. A. Rodríguez S., M. Socolovsky and J. L. Vázquez B, *Rev. Mex. Fís.* **34**, 452 (1988).

[79] B. Green, *Basics of CFT, string compactifications, mirror symmetry and space-time topology change*, preliminary notes. Summer School in High energy Physics and Cosmology, 12 June-28 July 1995, ICTP, Trieste, Italy.

[80] F. Mandel and G. Shaw, *Quantum Field theory*, John Wiley and Sons, (1991).

[81] P. K. Townsend, *Preprint hep-th 9612121v2*, (1996).

[82] H. J. de Vega and N. Sánchez, Phys. Lett. **B 197** (1987), 320;

[83] Proceedings of the Erice Schools: "String Quantum Gravity and Physics at the Planck Energy Scale", 21-28 June 1992, Edited by N. Sánchez, World Scientific, 1993. Edited by N. Sánchez and A. Zichichi, Kluwer Acad. Publ.
Third D. Chalonge School 'Current Topics in Astrofundamental Physics', 4-16 September 1994, Edited by N. Sánchez and A. Zichichi, NATO ASI Series Vol. 467, Kluwer Acad. Publ.

[84] L. P. Grishchuk in Proceedings of the Erice School: "String Gravity and Physics at the Planck Energy Scale", 8-19 September 1995, Edited by N. Sánchez, World Scientific, 1996. Edited by N. Sánchez and A. Zichichi, Kluwer Acad. Publ.

[85] A. Dabholkar and J. M. Quashnock, *Nucl. Phys.* **B 333**, 815 (1990). E. J. Copeland, D. Haws, and M. B. Hindmarsh, *Phys. Rev.* **D 42**, 726 (1990).

[86] D. Amati and C. Klimčik, *Phys. Lett.* **B 210**, 92 (1988). H. J. de Vega and N. Sánchez, *Nucl. Phys.* **B 317**, 706 (1989).

[87] A. Vilenkin and E. P. S. Shellard, *Cosmic Strings and Other Topological Defects* (CUP 1994).

[88] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products.* Edited by Alan Jeffrey. Academic Press Inc. (1980).

[89] M. Abramowitz and I. A. Stegun, *Pocketbook of Mathematical Functions*, Verlag Harri Deutsch, (1984).

[90] C. Kuo, *Preprint gr-qc 9611064v2*, (1996).

[91] H. Epstein, V. Glaser and A. Jaffe, *Nuovo Cimento* **36**, 1016, (1965).

[92] L. H. Ford and T. A. Roman, *Preprint gr-qc 9607003v2*, (1996).

[93] A. D. Helfer, *Preprint gr-qc 9612029*, (1996).

[94] H. Stephani, *General Relativity*, Cambridge University Press, (1990).

[95] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, W. H. Freeman & and Co. (1973).

[96] E. J. Copeland, A. Vázquez, and H. J. de Vega. *Preprint LPTHE-Paris95-26/SUSX-TH95-31, hep-th 960112*, (1996). (To be revised.)

[97] A. Vilenkin, *Phys. Rev.* **D 23**, 852 (1981).

[98] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, John Wiley and Sons, (1972).

[99] J. D. Jackson, *Classical Electrodynamics*, John Wiley and Sons, (1975).

[100] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, Pergamon Press, (1989).

[101] H. Goldstein, *Classical Mechanics*, Addison Wesley, (1980).

[102] B. F. Schutz, *A first course in general relativity*, Cambridge University Press, (1990).

[103] T.W. B. Kibble in 'The Formation and Evolution of Cosmic Strings'. Proceedings of a Workshop supported by the SERC. 3-7 July 1989. Edited by G. Gibbons, S. Hawking and T. Vachaspati. Cambridge University Press.

[104] N. Turok, *Phys. Lett.* **126B**, 437 (1983).

[105] W. A. Hiscock and J. B. Lail, *Phys. Rev.*. **D37**, 869, (1988).

[106] J. Silk and A. Vilenkin, *Phys. Rev. Lett.* **53**, 1700, (1984).

[107] M. E. V. Costa, and H. J. de Vega, *Ann. Phys.* **211**, 223 (1991).

[108] T. Dray, and G. 't Hooft, *Nucl. Phys.* **B253**, 173 (1985).

[109] C. O. Loustó and N. Sánchez, *Nucl. Phys.* **B355**, 231 (1991).

[110] C. O. Loustó and N. Sánchez, *Phys. Lett.*. **B232**, 462 (1989).

[111] P. C. Aichelburg and R. U. Sexl, J. Gen. Rel. Grav.2, 303 (1971).

APPENDICES

APPENDIX A

THE WEAK-FIELD METRIC $h^{\mu\nu}$ FROM A QUANTUM BOSONIC STRING

A.1 The metric components from the quantum bosonic string energy-momentum tensor in Minkowski space-time

In this section we will present the explicit results for the weak-field metric components of a quantum bosonic string. Most of the results presented in this appendix were performed using Maple V Release 3. From eqs.(5.12)-(5.15) and eqs.(5.8), (4.3)-(4.5) we find:

$$h_a^{00}(\vec{x},t) = \frac{4G}{r}\int d^3\vec{x}'\ \langle \hat{T}^{00}(r',t_{ret})\rangle \tag{A.1}$$

$$h_a^{00}(\vec{x},t) = \frac{16\pi\ G}{r}\int_0^R dr'\ r'\left[F(t_{ret}-r') - F(t_{ret}-r')\right] \tag{A.2}$$

we can now solve the remaining integral over r'[1] with the help of appendix B

$$\begin{aligned} h_a^{00}(\vec{x},t) = G\ \sqrt{\frac{2\alpha}{\pi}}\ &\left(\left(\Phi\left(\frac{1}{2}\frac{((t-r)-R)\sqrt{2}}{\sqrt{\alpha}}\right) - \Phi\left(\frac{1}{2}\frac{((t-r)+R)\sqrt{2}}{\sqrt{\alpha}}\right)\right)\times\right.\\ &\left((t-r)^2 - R^2 + \frac{2}{3}\alpha\right) + \left(\mathrm{e}^{\left(-1/2\,\frac{((t-r)+R)^2}{\alpha}\right)} + \mathrm{e}^{\left(-1/2\,\frac{((t-r)-R)^2}{\alpha}\right)}\right)\times\\ &\left.\sqrt{2}\sqrt{\frac{\alpha}{\pi}}\left(\frac{2}{3}R-(t-r)\right)\right)\Big/(\pi\,\alpha^2\,r) \end{aligned} \tag{A.3}$$

$$h_a^{11}(\vec{x},t) = \frac{4G}{r}\int d^3\vec{x}'\ \langle \hat{T}^{11}(r',t_{ret})\rangle \tag{A.4}$$

$$\begin{aligned} h_a^{11}(\vec{x},t) \quad = \quad & \frac{16\pi\ G}{r}\int_0^R dr'\ r'^2\left(-\frac{1}{r'^2}\left[H(t_{ret}-r') + H(t_{ret}-r')\right] + \right.\\ & \left.\frac{1}{r'^3}\left[E(t_{ret}-r') - E(t_{ret}-r')\right]\right) + \frac{4G}{r}\int_0^R dr'\int_0^{2\pi} d\phi\int_0^{\pi} d\gamma\ r'^2\cos^2\phi\sin^3\gamma\ \times \end{aligned}$$

[1] the final results in this appendix were performed using MapleV v.3, the output style in latex is that of maple.sty

$$\left(\frac{1}{r'}\left[F(t_{ret}-r')-F(t_{ret}-r')\right]+\frac{3}{r'^2}\left[H(t_{ret}-r')+H(t_{ret}-r')\right]\right.$$
$$\left.-\frac{1}{r'^3}\left[E(t_{ret}-r')-E(t_{ret}-r')\right]\right) \tag{A.5}$$

performing the integrals over α and γ we see that

$$h_a^{11}(\vec{x},t)=\frac{1}{3}h_a^{00}(\vec{x},t) \tag{A.6}$$

similarly one finds

$$h_a^{22}(\vec{x},t)=h_a^{22}(\vec{x},t)=\frac{1}{3}h_a^{00}(\vec{x},t)$$

Thus, we obtain:

$$h_a^{ii}(\vec{x},t)=\frac{G}{3}\sqrt{\frac{2\alpha}{\pi}}\left(\left(\Phi\left(\frac{1}{2}\frac{((t-r)-R)\sqrt{2}}{\sqrt{\alpha}}\right)-\Phi\left(\frac{1}{2}\frac{((t-r)+R)\sqrt{2}}{\sqrt{\alpha}}\right)\right)\right)\times$$
$$\left((t-r)^2-R^2+\frac{2}{3}\alpha\right)+\left(\mathrm{e}^{\left(-1/2\,\frac{((t-r)+R)^2}{\alpha}\right)}+\mathrm{e}^{\left(-1/2\,\frac{((t-r)-R)^2}{\alpha}\right)}\right)\times$$
$$\left.\sqrt{2}\sqrt{\frac{\alpha}{\pi}}\left(\frac{2}{3}R-(t-r)\right)\right)\Big/(\pi\,\alpha^2\,r) \tag{A.7}$$

$$h_a^{01}(\vec{x},t)=\frac{4G}{r}\int d^3\vec{x}'\;\langle\hat{T}^{01}(r',t_{ret})\rangle \tag{A.8}$$

$$h_a^{01}(\vec{x},t)=\frac{4G}{r}\int_0^R dr'\int_0^{2\pi}d\phi\int_0^{\pi}d\gamma\;r'^2\;\cos\phi\sin^2\gamma\;C(t_{ret},r') \tag{A.9}$$

$$h_a^{01}(\vec{x},t)=0 \tag{A.10}$$

Similarly one finds that

$$h_a^{03}(\vec{x},t)=h_a^{03}(\vec{x},t)=0.$$

For $h_b^{\mu\nu}(\vec{x},t)$ we have the following:

$$h_b^{00}(\vec{x},t)=\frac{4G\;x^i}{r^3}\int d^3\vec{x}'\;x'^i\;\langle\hat{T}^{00}(r',t_{ret})\rangle \tag{A.11}$$

$$\begin{aligned} h_b^{00}(\vec{x},t) \;&=\; \frac{4G}{r^2}\int_0^R dr'\int_0^{2\pi}d\phi\int_0^{\pi}d\gamma\;r'^3\;\langle\hat{T}^{00}(r',t_{ret})\rangle\times \\ &\left[\cos\phi\sin^2\gamma\cos\alpha_0\sin\gamma_0+\sin\phi\sin\gamma\sin\alpha_0\sin\gamma_0+\sin\gamma\cos\gamma\cos\gamma_0\right] \end{aligned} \tag{A.12}$$

thus, $h_b^{00}(\vec{x},t)=0$ similarly one finds $h_b^{ii}(\vec{x},t)=0$. For $h_b^{0i}(\vec{x},t)$ we have:

$$h_b^{0i}(\vec{x},t)=\frac{4G\;x^i}{r^3}\int d^3\vec{x}'\;x'^i\;\langle\hat{T}^{0i}(r',t_{ret})\rangle \tag{A.13}$$

$$h_b^{01}(\vec{x},t) = \frac{4G}{r^2}\cos\alpha_0\sin\gamma_0\int_0^R dr'\int_0^{2\pi}d\phi\int_0^{\pi}d\gamma\; r'^3\; cos^2\alpha\sin^3\gamma\; C(t_{ret},r') \tag{A.14}$$

$$h_b^{01}(\vec{x},t) = \frac{16\pi\; G}{3r^2}\cos\alpha_0\sin\gamma_0\int_0^R dr'\; r'^3\; C(t_{ret},r') \tag{A.15}$$

substituting the expression for $C(t_{ret},r')$ and performing the r' integration we obtain:

$$\begin{aligned} h_b^{01}(\vec{x},t) = \frac{1}{3}G\;\sqrt{\frac{2\alpha}{\pi}}\;\Bigg(&\left(\Phi\left(\frac{1}{2}\,\frac{(\,(t-r)+R\,)\sqrt{2}}{\sqrt{\alpha}}\right)-\Phi\left(\frac{1}{2}\,\frac{(\,(t-r)-R\,)\sqrt{2}}{\sqrt{\alpha}}\right)\right)\\ &(\,(t-r)\,R^2-(t-r)^3-2\,\alpha\,(t-r)\,)-\alpha\,\left(\mathrm{e}^{\left(-1/2\,\frac{(\,(t-r)+R\,)^2}{\alpha}\right)}-\mathrm{e}^{\left(-1/2\,\frac{(\,(t-r)-R\,)^2}{\alpha}\right)}\right)\times\\ &\sqrt{2}\sqrt{\frac{\alpha}{\pi}}+R\,(t-r)\,\left(\mathrm{e}^{\left(-1/2\,\frac{(\,(t-r)+R\,)^2}{\alpha}\right)}+\mathrm{e}^{\left(-1/2\,\frac{(\,(t-r)-R\,)^2}{\alpha}\right)}\right)\sqrt{2}\sqrt{\frac{\alpha}{\pi}}\Bigg)\times\\ &\cos(\,\alpha_0)\,)\sin(\,\gamma_0\,)\Big/(\,\pi\,\alpha^2r^2\,) \end{aligned} \tag{A.16}$$

Similarly we obtain:

$$\begin{aligned} h_b^{02}(\vec{x},t) = \frac{1}{3}G\;\sqrt{\frac{2\alpha}{\pi}}\;\Bigg(&\\ &\left(\Phi\left(\frac{1}{2}\,\frac{(\,(t-r)+R\,)\sqrt{2}}{\sqrt{\alpha}}\right)-\Phi\left(\frac{1}{2}\,\frac{(\,(t-r)-R\,)\sqrt{2}}{\sqrt{\alpha}}\right)\right)(\,(t-r)\,R^2-(t-r)^3-2\,\alpha\,(t-r)\,)\\ &-\alpha\,\left(\mathrm{e}^{\left(-1/2\,\frac{(\,(t-r)+R\,)^2}{\alpha}\right)}-\mathrm{e}^{\left(-1/2\,\frac{(\,(t-r)-R\,)^2}{\alpha}\right)}\right)\sqrt{2}\sqrt{\frac{\alpha}{\pi}}\\ &+R\,(t-r)\,\left(\mathrm{e}^{\left(-1/2\,\frac{(\,(t-r)+R\,)^2}{\alpha}\right)}+\mathrm{e}^{\left(-1/2\,\frac{(\,(t-r)-R\,)^2}{\alpha}\right)}\right)\sqrt{2}\sqrt{\frac{\alpha}{\pi}}\Bigg)\sin(\,\alpha_0\,)\sin(\,\gamma_0\,)\Big/(\,\pi\,\alpha^2r^2\,) \end{aligned} \tag{A.17}$$

$$\begin{aligned} h_b^{03}(\vec{x},t) = \frac{1}{3}G\;\sqrt{\frac{2\alpha}{\pi}}\;\Bigg(&\\ &\left(\Phi\left(\frac{1}{2}\,\frac{(\,(t-r)+R\,)\sqrt{2}}{\sqrt{\alpha}}\right)-\Phi\left(\frac{1}{2}\,\frac{(\,(t-r)-R\,)\sqrt{2}}{\sqrt{\alpha}}\right)\right)(\,(t-r)\,R^2-(t-r)^3-2\,\alpha\,(t-r)\,)\\ &-\alpha\,\left(\mathrm{e}^{\left(-1/2\,\frac{(\,(t-r)+R\,)^2}{\alpha}\right)}-\mathrm{e}^{\left(-1/2\,\frac{(\,(t-r)-R\,)^2}{\alpha}\right)}\right)\sqrt{2}\sqrt{\frac{\alpha}{\pi}}\\ &+R\,(t-r)\,\left(\mathrm{e}^{\left(-1/2\,\frac{(\,(t-r)+R\,)^2}{\alpha}\right)}+\mathrm{e}^{\left(-1/2\,\frac{(\,(t-r)-R\,)^2}{\alpha}\right)}\right)\sqrt{2}\sqrt{\frac{\alpha}{\pi}}\Bigg)\cos(\,\gamma_0\,)\Big/(\,\pi\,\alpha^2\,r^2\,) \end{aligned} \tag{A.18}$$

For $h_3^{\mu\nu}(\vec{x},t)$ we have the following:

$$h_c^{00}(\vec{x},t) = \frac{4G\;x^ix^j}{r^5}\int d^3\vec{x}'\;(3x'^ix'^j-r'^2\delta^{ij})\;\langle\hat{T}^{00}(r',t_{ret})\rangle \tag{A.19}$$

$$h_c^{00}(\vec{x},t) \;\;=\;\; \frac{4G}{r^3}\int_0^R dr'\int_0^{2\pi}d\phi\int_0^{\pi}d\gamma\; r'^4\;\sin\gamma\;\langle\hat{T}^{00}(r',t_{ret})\rangle\times$$

$$\Big[3(\cos^2\phi\sin^2\gamma\cos_0^\alpha\sin^2\gamma_0+\sin^2\phi\sin^2\gamma\sin^2\alpha_0\sin^2\gamma_0+\cos^2\gamma\cos^2\gamma_0)-1\Big] \tag{A.20}$$

performing the remaining ϕ and γ integrals we find that

$$h_c^{00}(\vec{x},t)=0. \tag{A.21}$$

$$h_c^{11}(\vec{x},t)=\frac{4G\,x^i x^j}{r^5}\int d^3\vec{x}'\,(3x'^i x'^j-r'^2\delta^{ij})\,\langle\hat{T}^{11}(r',t_{ret})\rangle \tag{A.22}$$

$$\begin{aligned}h_c^{11}(\vec{x},t) \;=\; & \frac{4G}{r^3}\int_0^R dr'\int_0^{2\pi}d\phi\int_0^\pi d\gamma\, r'^4\;\cos^2\phi\sin^3\gamma\,\langle\hat{T}^{00}(r',t_{ret})\rangle\times\\ & \Big[3(\cos^2\phi\sin^2\gamma\cos_0^\alpha\sin^2\gamma_0+\sin^2\phi\sin^2\gamma\sin^2\alpha_0\sin^2\gamma_0+\cos^2\gamma\cos^2\gamma_0)-1\Big]\times\\ & \Big(\frac{1}{r'}\left[F(t_{ret}+r')-F(t_{ret}-r')\right]+\frac{3}{r'^2}\left[H(t_{ret}+r')+H(t_{ret}-r')\right]\\ & -\frac{3}{r'^3}\left[E(t_{ret}+r')-E(t_{ret}-r')\right]\Big)\end{aligned} \tag{A.23}$$

performing the remaining ϕ and γ integrals we find:

$$\begin{aligned}h_c^{11}(\vec{x},t) \;=\; & \frac{32\pi\,G}{15r^3}\Big[3\cos^2\alpha_0\sin^2\gamma_0-1\Big]\int_0^R dr'\,r'^4\,\Big(\frac{1}{r'}\left[F(t_{ret}+r')-F(t_{ret}-r')\right]+\\ & \frac{3}{r'^2}\left[H(t_{ret}+r')+H(t_{ret}-r')\right]-\frac{3}{r'^3}\left[E(t_{ret}+r')-E(t_{ret}-r')\right]\Big)\end{aligned} \tag{A.24}$$

performing the r' integration we obtain:

$$\begin{aligned}h_c^{11}(\vec{x},t)=\frac{1}{30}\Big(3\cos(\,\alpha_0\,)^2\sin(\,\gamma_0\,)^2-1\Big)\;G\;\sqrt{\frac{2\alpha}{\pi}}\;\Bigg(\\ \left(\Phi\left(\frac{1}{2}\,\frac{(\,(t-r)+R\,)\sqrt{2}}{\sqrt{\alpha}}\right)-\Phi\left(\frac{1}{2}\,\frac{(\,(t-r)-R\,)\sqrt{2}}{\sqrt{\alpha}}\right)\right)\\ (\,-R^4-5\,(t-r)^4-5\,\alpha^2-20\,\alpha\,(t-r)^2+6\,(t-r)^2\,R^2-4\,\alpha\,R^2\,)\\ +(t-r)\sqrt{2}\sqrt{\frac{\alpha}{\pi}}\,(\,3\,(t-r)^2+R^2-3\,\alpha\,)\;\left(\mathrm{e}^{\left(-1/2\,\frac{(\,(t-r)+R\,)^2}{\alpha}\right)}-\mathrm{e}^{\left(-1/2\,\frac{(\,(t-r)-R\,)^2}{\alpha}\right)}\right)\\ +R\sqrt{2}\sqrt{\frac{\alpha}{\pi}}\;\left(9\,(t-r)^2+\frac{11}{3}\,R^2+5\,\alpha\right)\;\left(\mathrm{e}^{\left(-1/2\,\frac{(\,(t-r)+R\,)^2}{\alpha}\right)}+\mathrm{e}^{\left(-1/2\,\frac{(\,(t-r)-R\,)^2}{\alpha}\right)}\right)\Bigg)\Big/(\\ \pi\,\alpha^2\,r^3\,)\end{aligned} \tag{A.25}$$

Similarly we find:

$$\begin{aligned}h_c^{22}(\vec{x},t)=\frac{1}{30}\Big(3\sin(\,\alpha_0\,)^2\sin(\,\gamma_0\,)^2-1\Big)\;G\;\sqrt{\frac{2\alpha}{\pi}}\;\Bigg(\\ \left(\Phi\left(\frac{1}{2}\,\frac{(\,(t-r)+R\,)\sqrt{2}}{\sqrt{\alpha}}\right)-\Phi\left(\frac{1}{2}\,\frac{(\,(t-r)-R\,)\sqrt{2}}{\sqrt{\alpha}}\right)\right)\\ (\,-R^4-5\,(t-r)^4-5\,\alpha^2-20\,\alpha\,(t-r)^2+6\,(t-r)^2\,R^2-4\,\alpha\,R^2\,)\end{aligned}$$

$$+ (t-r)\sqrt{2}\sqrt{\frac{\alpha}{\pi}}\,(3\,(t-r)^2+R^2-3\,\alpha)\,\left(e^{\left(-1/2\,\frac{((t-r)+R)^2}{\alpha}\right)}-e^{\left(-1/2\,\frac{((t-r)-R)^2}{\alpha}\right)}\right)$$

$$+ R\sqrt{2}\sqrt{\frac{\alpha}{\pi}}\,\left(9\,(t-r)^2+\frac{11}{3}\,R^2+5\,\alpha\right)\,\left(e^{\left(-1/2\,\frac{((t-r)+R)^2}{\alpha}\right)}+e^{\left(-1/2\,\frac{((t-r)-R)^2}{\alpha}\right)}\right)\Bigg)\Big/\Big(\pi\,\alpha^2\,r^3\,\Big) \tag{A.26}$$

$$h_c^{33}(\vec{x},t)=\frac{1}{30}\left(2-3\sin(\gamma_0)^2\right)\,G\,\sqrt{\frac{2\alpha}{\pi}}\,\Bigg(\left(\Phi\left(\frac{1}{2}\,\frac{((t-r)+R)\sqrt{2}}{\sqrt{\alpha}}\right)-\Phi\left(\frac{1}{2}\,\frac{((t-r)-R)\sqrt{2}}{\sqrt{\alpha}}\right)\right)$$

$$(-R^4-5\,(t-r)^4-5\,\alpha^2-20\,\alpha\,(t-r)^2+6\,(t-r)^2\,R^2-4\,\alpha\,R^2\,)$$

$$+ (t-r)\sqrt{2}\sqrt{\frac{\alpha}{\pi}}\,(3\,(t-r)^2+R^2-3\,\alpha)\,\left(e^{\left(-1/2\,\frac{((t-r)+R)^2}{\alpha}\right)}-e^{\left(-1/2\,\frac{((t-r)-R)^2}{\alpha}\right)}\right)$$

$$+ R\sqrt{2}\sqrt{\frac{\alpha}{\pi}}\,\left(9\,(t-r)^2+\frac{11}{3}\,R^2+5\,\alpha\right)\,\left(e^{\left(-1/2\,\frac{((t-r)+R)^2}{\alpha}\right)}+e^{\left(-1/2\,\frac{((t-r)-R)^2}{\alpha}\right)}\right)\Bigg)\Big/\Big(\pi\,\alpha^2\,r^3\,\Big) \tag{A.27}$$

$$h_c^{01}(\vec{x},t)=\frac{4G\;x^ix^j}{r^5}\int d^3\vec{x}'\;(3x'^ix'^j-r'^2\delta^{ij})\;\langle\hat{T}^{01}(r',t_{ret})\rangle \tag{A.28}$$

$$\begin{aligned} h_c^{01}(\vec{x},t) \;&=\; \frac{4G}{r^3}\int_0^R dr'\int_0^{2\pi}d\phi\int_0^{\pi}d\gamma\;r'^4\cos\phi\sin^2\gamma\;C(r',t_{ret})\times \\ &\Big[3(\cos^2\phi\sin^2\gamma\cos_0^{\alpha}\sin^2\gamma_0+\sin^2\phi\sin^2\gamma\sin^2\alpha_0\sin^2\gamma_0+\cos^2\gamma\cos^2\gamma_0)-1\Big] \end{aligned} \tag{A.29}$$

performing the ϕ and γ integrals we find:

$$h_c^{01}(\vec{x},t)=0 \tag{A.30}$$

similarly one finds that

$$h_c^{02}(\vec{x},t)=h_c^{03}(\vec{x},t)=0$$

$$h_d^{00}(\vec{x},t)=\frac{4G\;x^ix^j}{r^3}\int d^3\vec{x}'\;x'^ix'^j\;\langle\hat{T}^{00}(r',t_{ret})\rangle \tag{A.31}$$

$$\begin{aligned} h_d^{00}(\vec{x},t) \;&=\; \frac{4G}{r}\int_0^R dr'\int_0^{2\pi}d\phi\int_0^{\pi}d\gamma\;r'^4\;\sin\gamma\;\langle\hat{T}^{00}(r',t_{ret})\rangle\times \\ &\Big[\cos^2\phi\sin^2\gamma\cos\alpha_0\sin^2\gamma_0+\sin^2\phi\sin^2\gamma\sin\alpha_0\sin^2\gamma_0+\cos^2\gamma\cos^2\gamma_0\Big] \end{aligned} \tag{A.32}$$

performing the ϕ and γ integrals we obtain:

$$h_d^{00}(\vec{x},t) = \frac{16\pi\ G}{3r}\int_0^R dr'\ r'^3\left[F(t_{ret}+r') - F(t_{ret}-r')\right] \tag{A.33}$$

$$h_d^{11}(\vec{x},t) = \frac{4G\ x^i x^j}{r^3}\int d^3\vec{x}'\ x'^i x'^j\ \langle \hat{T}^{11}(r',t_{ret})\rangle \tag{A.34}$$

$$\begin{aligned} h_d^{11}(\vec{x},t) \quad = \quad & \frac{4G}{r}\int_0^R dr'\int_0^{2\pi} d\phi\int_0^{\pi} d\gamma\ r'^4\ \cos^2\phi\sin^3\gamma\ \times \\ & \left[\cos^2\phi\sin^2\gamma\cos\alpha_0\sin^2\gamma_0 + \sin^2\phi\sin^2\gamma\sin\alpha_0\sin^2\gamma_0 + \cos^2\gamma\cos^2\gamma_0\right]\times \\ & \left(\frac{1}{r'}\left[F(t_{ret}+r') - F(t_{ret}+r')\right] + \frac{3}{r'^2}\left[H(t_{ret}+r') + H(t_{ret}+r')\right]\right. \\ & \left. - \frac{3}{r'^3}\left[E(t_{ret}+r') - E(t_{ret}+r')\right]\right) + \\ & \frac{16\pi\ G}{3r}\int_0^R dr'\ r'^4\left(-\frac{1}{r'^2}\left[H(t_{ret}+r') + H(t_{ret}+r')\right] + \right. \\ & \left.\frac{1}{r'^3}\left[E(t_{ret}+r') - E(t_{ret}+r')\right]\right) \end{aligned} \tag{A.35}$$

performing the remaining ϕ and γ integrals

$$\begin{aligned} h_d^{11}(\vec{x},t) \quad = \quad & \frac{16\pi\ G}{15r}\left[2\cos^2\alpha_0\sin^2\gamma_0 + 1\right]\int_0^R dr'\ \left(\frac{1}{r'}\left[F(t_{ret}+r') - F(t_{ret}+r')\right] + \right. \\ & \left.\frac{3}{r'^2}\left[H(t_{ret}+r') + H(t_{ret}+r')\right] - \frac{3}{r'^3}\left[E(t_{ret}+r') - E(t_{ret}+r')\right]\right) + \\ & \frac{16\pi\ G}{3r}\int_0^R dr'\ r'^4\left(-\frac{1}{r'^2}\left[H(t_{ret}+r') + H(t_{ret}+r')\right] + \right. \\ & \left.\frac{1}{r'^3}\left[E(t_{ret}+r') - E(t_{ret}+r')\right]\right) \end{aligned} \tag{A.36}$$

Similarly one finds:

$$\begin{aligned} h_d^{22}(\vec{x},t) \quad = \quad & \frac{16\pi\ G}{15r}\left[2\sin^2\alpha_0\sin^2\gamma_0 + 1\right]\int_0^R dr'\ \left(\frac{1}{r'}\left[F(t_{ret}+r') - F(t_{ret}+r')\right] + \right. \\ & \left.\frac{3}{r'^2}\left[H(t_{ret}+r') + H(t_{ret}+r')\right] - \frac{3}{r'^3}\left[E(t_{ret}+r') - E(t_{ret}+r')\right]\right) + \\ & \frac{16\pi\ G}{3r}\int_0^R dr'\ r'^4\left(-\frac{1}{r'^2}\left[H(t_{ret}+r') + H(t_{ret}+r')\right] + \right. \\ & \left.\frac{1}{r'^3}\left[E(t_{ret}+r') - E(t_{ret}+r')\right]\right) \end{aligned} \tag{A.37}$$

$$h_d^{22}(\vec{x},t) \quad = \quad \frac{16\pi\ G}{5r}\left[1 - \frac{2}{3}\sin^2\gamma_0\right]\int_0^R dr'\ \left(\frac{1}{r'}\left[F(t_{ret}+r') - F(t_{ret}+r')\right] + \right.$$

$$\frac{3}{r'^2}\left[H(t_{ret}+r')+H(t_{ret}+r')\right]-\frac{3}{r'^3}\left[E(t_{ret}+r')-E(t_{ret}+r')\right]\Big)+$$
$$\frac{16\pi\, G}{3r}\int_0^R dr'\, r'^4\left(-\frac{1}{r'^2}\left[H(t_{ret}+r')+H(t_{ret}+r')\right]+\right.$$
$$\left.\frac{1}{r'^3}\left[E(t_{ret}+r')-E(t_{ret}+r')\right]\right) \tag{A.38}$$

$$h_d^{01}(\vec{x},t)=\frac{4G\,x^i x^j}{r^3}\int d^3\vec{x}'\;x'^i x'^j\;\langle\hat{T}^{01}(r',t_{ret})\rangle \tag{A.39}$$

$$\begin{aligned}h_d^{01}(\vec{x},t) &= \frac{4G}{r}\int_0^R dr'\int_0^{2\pi}d\phi\int_0^{\pi}d\gamma\; r'^4\;\cos\phi\sin^2\gamma\; C(r',t_{ret})\times\\ &\Big[\cos^2\phi\sin^2\gamma\cos\alpha_0\sin^2\gamma_0+2\cos\phi\sin\phi\sin^2\gamma\cos\alpha_0\sin\alpha_0\sin^2\gamma_0+\\ &\sin^2\phi\sin^2\gamma\sin\alpha_0\sin^2\gamma_0+\cos^2\gamma\cos^2\gamma_0+\\ &2\cos\phi\sin\gamma\cos\gamma\cos\alpha_0\sin\gamma_0\cos\gamma_0+2\sin\phi\sin\gamma\cos\gamma\sin\alpha_0\sin\gamma_0\cos\gamma_0\Big]\end{aligned} \tag{A.40}$$

performing the ϕ and γ integrals we find that

$$h_d^{01}(\vec{x},t)=0 \tag{A.41}$$

similarly one finds:

$$h_d^{02}(\vec{x},t)=h_d^{03}(\vec{x},t)=0.$$

The other terms in the weak-field metric can be obtained from eqs.(A.2)-(A.41) with the results:

$$\begin{aligned}\dot{h}_b^{01}(\vec{x},t) &= \frac{1}{3}G\,\sqrt{\frac{2\alpha}{\pi}}\;\Bigg(\left(\frac{\%2\,\sqrt{2}}{\sqrt{\pi}\,\sqrt{\alpha}}-\frac{\%1\,\sqrt{2}}{\sqrt{\pi}\,\sqrt{\alpha}}\right)\left(\,(t-r)\,R^2-(t-r)^3-2\,\alpha\,(t-r)\,\right)\\ &+\left(\Phi\left(\frac{1}{2}\,\frac{(\,(t-r)+R\,)\,\sqrt{2}}{\sqrt{\alpha}}\right)-\Phi\left(\frac{1}{2}\,\frac{(\,(t-r)-R\,)\,\sqrt{2}}{\sqrt{\alpha}}\right)\right)\left(\,R^2-3\,(t-r)^2-2\,\alpha\,\right)\\ &-\alpha\left(-\frac{(\,t+R\,)\,\%2}{\alpha}+\frac{(\,(t-r)-R\,)\,\%1}{\alpha}\right)\sqrt{2}\,\sqrt{\frac{\alpha}{\pi}}+R\,(\,\%2+\%1\,)\,\sqrt{2}\,\sqrt{\frac{\alpha}{\pi}}\\ &+R\,(t-r)\left(-\frac{(\,(t-r)+R\,)\,\%2}{\alpha}-\frac{(\,(t-r)-R\,)\,\%1}{\alpha}\right)\sqrt{2}\,\sqrt{\frac{\alpha}{\pi}}\Bigg)\cos(\,\alpha_0\,)\sin(\,\gamma_0\,)\Big/(\,r\,\pi\\ &\alpha^2\,)\end{aligned} \tag{A.42}$$

$$\begin{aligned}\dot{h}_b^{02}(\vec{x},t) &= \frac{1}{3}G\,\sqrt{\frac{2\alpha}{\pi}}\;\Bigg(\left(\frac{\%2\,\sqrt{2}}{\sqrt{\pi}\,\sqrt{\alpha}}-\frac{\%1\,\sqrt{2}}{\sqrt{\pi}\,\sqrt{\alpha}}\right)\left(\,(t-r)\,R^2-(t-r)^3-2\,\alpha\,(t-r)\,\right)\\ &+\left(\Phi\left(\frac{1}{2}\,\frac{(\,(t-r)+R\,)\,\sqrt{2}}{\sqrt{\alpha}}\right)-\Phi\left(\frac{1}{2}\,\frac{(\,(t-r)-R\,)\,\sqrt{2}}{\sqrt{\alpha}}\right)\right)\left(\,R^2-3\,(t-r)^2-2\,\alpha\,\right)\\ &-\alpha\left(-\frac{(\,t+R\,)\,\%2}{\alpha}+\frac{(\,(t-r)-R\,)\,\%1}{\alpha}\right)\sqrt{2}\,\sqrt{\frac{\alpha}{\pi}}+R\,(\,\%2+\%1\,)\,\sqrt{2}\,\sqrt{\frac{\alpha}{\pi}}\end{aligned}$$

$$+ R\,(t-r)\,\left(-\frac{(\,(t-r)+R\,)\,\%2}{\alpha} - \frac{(\,(t-r)-R\,)\,\%1}{\alpha}\right)\,\sqrt{2}\,\sqrt{\frac{\alpha}{\pi}}\right)\sin(\,\alpha_0\,)\sin(\,\gamma_0\,)\Big/(\,r\,\pi\,\alpha^2\,) \tag{A.43}$$

$$\dot{h}_b^{03}(\vec{x},t) = \frac{1}{3}G\,\sqrt{\frac{2\alpha}{\pi}}\,\left(\left(\frac{\%2\,\sqrt{2}}{\sqrt{\pi}\,\sqrt{\alpha}} - \frac{\%1\,\sqrt{2}}{\sqrt{\pi}\,\sqrt{\alpha}}\right)(\,(t-r)\,R^2-(t-r)^3-2\,\alpha\,(t-r)\,)\right.$$
$$+\left(\Phi\left(\frac{1}{2}\,\frac{(\,(t-r)+R\,)\,\sqrt{2}}{\sqrt{\alpha}}\right) - \Phi\left(\frac{1}{2}\,\frac{(\,(t-r)-R\,)\,\sqrt{2}}{\sqrt{\alpha}}\right)\right)\,(\,R^2-3\,(t-r)^2-2\,\alpha\,)$$
$$-\,\alpha\,\left(-\frac{(\,t+R\,)\,\%2}{\alpha} + \frac{(\,(t-r)-R\,)\,\%1}{\alpha}\right)\,\sqrt{2}\,\sqrt{\frac{\alpha}{\pi}} + R\,(\,\%2+\%1\,)\,\sqrt{2}\,\sqrt{\frac{\alpha}{\pi}}$$
$$\left.+ R\,(t-r)\,\left(-\frac{(\,(t-r)+R\,)\,\%2}{\alpha} - \frac{(\,(t-r)-R\,)\,\%1}{\alpha}\right)\,\sqrt{2}\,\sqrt{\frac{\alpha}{\pi}}\right)\cos(\,\gamma_0\,)\Big/(\,r\,\pi\,\alpha^2\,) \tag{A.44}$$

$$\dot{h}_c^{11}(\vec{x},t) = \frac{1}{60}\left(3\cos(\,\alpha_0\,)^2\sin(\,\gamma_0\,)^2-1\right)\,G\,\sqrt{\frac{2\alpha}{\pi}}\,\Bigg($$
$$\left(\frac{\%2\,\sqrt{2}}{\sqrt{\pi}\,\sqrt{\alpha}} - \frac{\%1\,\sqrt{2}}{\sqrt{\pi}\,\sqrt{\alpha}}\right)\,(\,-R^4-5\,(t-r)^4-5\,\alpha^2-20\,\alpha\,(t-r)^2+6\,(t-r)^2\,R^2-4\,\alpha\,R^2\,)\,+$$
$$\left(\Phi\left(\frac{1}{2}\,\frac{(\,(t-r)+R\,)\,\sqrt{2}}{\sqrt{\alpha}}\right) - \Phi\left(\frac{1}{2}\,\frac{(\,(t-r)-R\,)\,\sqrt{2}}{\sqrt{\alpha}}\right)\right)$$
$$(\,-20\,(t-r)^3-40\,\alpha\,(t-r)+12\,(t-r)\,R^2\,) + \sqrt{2}\,\sqrt{\frac{\alpha}{\pi}}\,(\,3\,(t-r)^2+R^2-3\,\alpha\,)\,(\,\%2-\%1\,)$$
$$+\,6\,(t-r)^2\,\sqrt{2}\,\sqrt{\frac{\alpha}{\pi}}\,(\,\%2-\%1\,)$$
$$+\,(t-r)\,\sqrt{2}\,\sqrt{\frac{\alpha}{\pi}}\,(\,3\,(t-r)^2+R^2-3\,\alpha\,)\,\left(-\frac{(\,(t-r)+R\,)\,\%2}{\alpha} + \frac{(\,(t-r)-R\,)\,\%1}{\alpha}\right)$$
$$+\,18\,R\,(t-r)\,(\,\%2+\%1\,)\,\sqrt{2}\,\sqrt{\frac{\alpha}{\pi}}$$
$$+\,R\,\sqrt{2}\,\sqrt{\frac{\alpha}{\pi}}\,\left(9\,(t-r)^2+\frac{11}{3}\,R^2+5\,\alpha\right)\,\left(-\frac{(\,(t-r)+R\,)\,\%2}{\alpha} - \frac{(\,(t-r)-R\,)\,\%1}{\alpha}\right)\Bigg)\Big/(\,r^2\,\pi\,\alpha^2\,) \tag{A.45}$$

$$\dot{h}_c^{22}(\vec{x},t) = \frac{1}{60}\left(3\sin(\,\alpha_0\,)^2\sin(\,\gamma_0\,)^2-1\right)\,G\,\sqrt{\frac{2\alpha}{\pi}}\,\Bigg($$
$$\left(\frac{\%2\,\sqrt{2}}{\sqrt{\pi}\,\sqrt{\alpha}} - \frac{\%1\,\sqrt{2}}{\sqrt{\pi}\,\sqrt{\alpha}}\right)\,(\,-R^4-5\,(t-r)^4-5\,\alpha^2-20\,\alpha\,(t-r)^2+6\,(t-r)^2\,R^2-4\,\alpha\,R^2\,)\,+$$
$$\left(\Phi\left(\frac{1}{2}\,\frac{(\,(t-r)+R\,)\,\sqrt{2}}{\sqrt{\alpha}}\right) - \Phi\left(\frac{1}{2}\,\frac{(\,(t-r)-R\,)\,\sqrt{2}}{\sqrt{\alpha}}\right)\right)$$
$$(\,-20\,(t-r)^3-40\,\alpha\,(t-r)+12\,(t-r)\,R^2\,) + \sqrt{2}\,\sqrt{\frac{\alpha}{\pi}}\,(\,3\,(t-r)^2+R^2-3\,\alpha\,)\,(\,\%2-\%1\,)$$
$$+\,6\,(t-r)^2\,\sqrt{2}\,\sqrt{\frac{\alpha}{\pi}}\,(\,\%2-\%1\,)$$

$$+ (t-r)\sqrt{2}\sqrt{\frac{\alpha}{\pi}}\,(\,3\,(t-r)^2 + R^2 - 3\,\alpha\,)\,\left(-\frac{(\,(t-r)+R\,)\,\%2}{\alpha} + \frac{(\,(t-r)-R\,)\,\%1}{\alpha}\right)$$

$$+ 18\,R\,(t-r)\,(\,\%2 + \%1\,)\sqrt{2}\sqrt{\frac{\alpha}{\pi}}$$

$$+ R\sqrt{2}\sqrt{\frac{\alpha}{\pi}}\left(9\,(t-r)^2 + \frac{11}{3}R^2 + 5\,\alpha\right)\left(-\frac{(\,(t-r)+R\,)\,\%2}{\alpha} - \frac{(\,(t-r)-R\,)\,\%1}{\alpha}\right)\Bigg)\Big/(\,r^2\,\pi\,\alpha^2\,) \tag{A.46}$$

$$\dot{h}_c^{33}(\vec{x},t) = \frac{1}{60}\left(2 - 3\sin(\,\gamma_0\,)^2\right) G\,\sqrt{\frac{2\alpha}{\pi}}\,\Bigg($$

$$\left(\frac{\%2\sqrt{2}}{\sqrt{\pi}\sqrt{\alpha}} - \frac{\%1\sqrt{2}}{\sqrt{\pi}\sqrt{\alpha}}\right)(\,-R^4 - 5\,(t-r)^4 - 5\,\alpha^2 - 20\,\alpha\,(t-r)^2 + 6\,(t-r)^2 R^2 - 4\,\alpha\,R^2\,) +$$

$$\left(\Phi\left(\frac{1}{2}\,\frac{(\,(t-r)+R\,)\sqrt{2}}{\sqrt{\alpha}}\right) - \Phi\left(\frac{1}{2}\,\frac{(\,(t-r)-R\,)\sqrt{2}}{\sqrt{\alpha}}\right)\right)$$

$$(\,-20\,(t-r)^3 - 40\,\alpha\,(t-r) + 12\,(t-r)\,R^2\,) + \sqrt{2}\sqrt{\frac{\alpha}{\pi}}\,(\,3\,(t-r)^2 + R^2 - 3\,\alpha\,)\,(\,\%2 - \%1\,)$$

$$+ 6\,(t-r)^2\sqrt{2}\sqrt{\frac{\alpha}{\pi}}\,(\,\%2 - \%1\,)$$

$$+ (t-r)\sqrt{2}\sqrt{\frac{\alpha}{\pi}}\,(\,3\,(t-r)^2 + R^2 - 3\,\alpha\,)\,\left(-\frac{(\,(t-r)+R\,)\,\%2}{\alpha} + \frac{(\,(t-r)-R\,)\,\%1}{\alpha}\right)$$

$$+ 18\,R\,(t-r)\,(\,\%2 + \%1\,)\sqrt{2}\sqrt{\frac{\alpha}{\pi}}$$

$$+ R\sqrt{2}\sqrt{\frac{\alpha}{\pi}}\left(9\,(t-r)^2 + \frac{11}{3}R^2 + 5\,\alpha\right)\left(-\frac{(\,(t-r)+R\,)\,\%2}{\alpha} - \frac{(\,(t-r)-R\,)\,\%1}{\alpha}\right)\Bigg)\Big/(\,r^2\,\pi\,\alpha^2\,) \tag{A.47}$$

$$\ddot{h}_d^{00}(\vec{x},t) = \frac{1}{12}G\,\sqrt{\frac{2\alpha}{\pi}}\,\Bigg($$

$$\pi\left(-\frac{(\,(t-r)+R\,)\,\%2\sqrt{2}}{\sqrt{\pi}\,\alpha^{3/2}} + \frac{(\,(t-r)-R\,)\,\%1\sqrt{2}}{\sqrt{\pi}\,\alpha^{3/2}}\right)(\,R^4 - (t-r)^4 - \alpha^2 - 4\,\alpha\,(t-r)^2\,)$$

$$+ 2\,\pi\left(\frac{\%2\sqrt{2}}{\sqrt{\pi}\sqrt{\alpha}} - \frac{\%1\sqrt{2}}{\sqrt{\pi}\sqrt{\alpha}}\right)(\,-4\,(t-r)^3 - 8\,\alpha\,(t-r)\,)$$

$$+ \pi\left(\Phi\left(\frac{1}{2}\,\frac{(\,(t-r)+R\,)\sqrt{2}}{\sqrt{\alpha}}\right) - \Phi\left(\frac{1}{2}\,\frac{(\,(t-r)-R\,)\sqrt{2}}{\sqrt{\alpha}}\right)\right)(\,-12\,(t-r)^2 - 8\,\alpha\,)$$

$$+ 2\,\alpha\left(-\frac{(\,(t-r)+R\,)\,\%2}{\alpha} + \frac{(\,(t-r)-R\,)\,\%1}{\alpha}\right)\sqrt{2}\sqrt{\frac{\pi}{\alpha}}\,(\,(t-r)^2 - R^2 - 3\,\alpha\,)$$

$$+ 6\,\alpha\,(\,\%2 - \%1\,)\sqrt{2}\sqrt{\frac{\pi}{\alpha}}\,(t-r) + \alpha\,(t-r)$$

$$\left(-\frac{\%2}{\alpha} + \frac{(\,(t-r)+R\,)^2\,\%2}{\alpha^2} + \frac{\%1}{\alpha} - \frac{(\,(t-r)-R\,)^2\,\%1}{\alpha^2}\right)\sqrt{2}\sqrt{\frac{\pi}{\alpha}}\,(\,(t-r)^2 - R^2 - 3\,\alpha\,)$$

$$+ 4\,\alpha\,(t-r)^2\left(-\frac{(\,(t-r)+R\,)\,\%2}{\alpha} + \frac{(\,(t-r)-R\,)\,\%1}{\alpha}\right)\sqrt{2}\sqrt{\frac{\pi}{\alpha}} + \alpha\,R$$

$$\left(-\frac{\%2}{\alpha}+\frac{(\,(t-r)+R\,)^2\,\%2}{\alpha^2}-\frac{\%1}{\alpha}+\frac{(\,(t-r)-R\,)^2\,\%1}{\alpha^2}\right)\sqrt{2}\,\sqrt{\frac{\pi}{\alpha}}$$
$$\left((t-r)^2+\frac{1}{3}\,R^2+\alpha\right)+4\,\alpha\,R\,\left(-\frac{(\,(t-r)+R\,)\,\%2}{\alpha}-\frac{(\,(t-r)-R\,)\,\%1}{\alpha}\right)\sqrt{2}\,\sqrt{\frac{\pi}{\alpha}}\,(t-r)$$
$$+\,2\,\alpha\,R\,(\,\%2+\%1\,)\,\sqrt{2}\,\sqrt{\frac{\pi}{\alpha}}\Bigg)\Big/\left(\pi^2\,\alpha^2\,r\right) \tag{A.48}$$

$$\ddot{h}_d^{11}(\vec{x},t)=\frac{1}{60}G\,\sqrt{\frac{2\alpha}{\pi}}\,\Bigg(\Bigg(\left(-\frac{(\,(t-r)+R\,)\,\%2\,\sqrt{2}}{\sqrt{\pi}\,\alpha^{3/2}}+\frac{(\,(t-r)-R\,)\,\%1\,\sqrt{2}}{\sqrt{\pi}\,\alpha^{3/2}}\right)$$
$$(\,-R^4-5\,(t-r)^4-5\,\alpha^2-20\,\alpha\,(t-r)^2+6\,(t-r)^2\,R^2-4\,\alpha\,R^2\,)$$
$$+\,2\,\left(\frac{\%2\,\sqrt{2}}{\sqrt{\pi}\,\sqrt{\alpha}}-\frac{\%1\,\sqrt{2}}{\sqrt{\pi}\,\sqrt{\alpha}}\right)(\,-20\,(t-r)^3-40\,\alpha\,(t-r)+12\,(t-r)\,R^2\,)\,+$$
$$\left(\Phi\left(\frac{1}{2}\,\frac{(\,(t-r)+R\,)\,\sqrt{2}}{\sqrt{\alpha}}\right)-\Phi\left(\frac{1}{2}\,\frac{(\,(t-r)-R\,)\,\sqrt{2}}{\sqrt{\alpha}}\right)\right)(\,-60\,(t-r)^2-40\,\alpha+12\,R^2\,)$$
$$+\,18\,\sqrt{2}\,\sqrt{\frac{\alpha}{\pi}}\,(t-r)\,(\,\%2-\%1\,)+2\,\sqrt{2}\,\sqrt{\frac{\alpha}{\pi}}\,(\,3\,(t-r)^2+R^2-3\,\alpha\,)\,\%3$$
$$+\,12\,(t-r)^2\,\sqrt{2}\,\sqrt{\frac{\alpha}{\pi}}\,\%3+(t-r)\,\sqrt{2}\,\sqrt{\frac{\alpha}{\pi}}\,(\,3\,(t-r)^2+R^2-3\,\alpha\,)$$
$$\left(-\frac{\%2}{\alpha}+\frac{(\,(t-r)+R\,)^2\,\%2}{\alpha^2}+\frac{\%1}{\alpha}-\frac{(\,(t-r)-R\,)^2\,\%1}{\alpha^2}\right)$$
$$+\,18\,R\,(\,\%2+\%1\,)\,\sqrt{2}\,\sqrt{\frac{\alpha}{\pi}}$$
$$+\,36\,R\,(t-r)\,\left(-\frac{(\,(t-r)+R\,)\,\%2}{\alpha}-\frac{(\,(t-r)-R\,)\,\%1}{\alpha}\right)\sqrt{2}\,\sqrt{\frac{\alpha}{\pi}}+R\,\sqrt{2}\,\sqrt{\frac{\alpha}{\pi}}$$
$$\left(9\,(t-r)^2+\frac{11}{3}\,R^2+5\,\alpha\right)\left(-\frac{\%2}{\alpha}+\frac{(\,(t-r)+R\,)^2\,\%2}{\alpha^2}-\frac{\%1}{\alpha}+\frac{(\,(t-r)-R\,)^2\,\%1}{\alpha^2}\right)\Bigg)$$
$$\left(\cos(\,\alpha_0\,)^2\,\sin(\,\gamma_0\,)^2-1\right)\Big/(\,r\,\pi\,\alpha^2\,)+\frac{1}{36}G\,\sqrt{\frac{2\alpha}{\pi}}\,\Bigg($$
$$\pi\,\left(-\frac{(\,(t-r)+R\,)\,\%2\,\sqrt{2}}{\sqrt{\pi}\,\alpha^{3/2}}+\frac{(\,(t-r)-R\,)\,\%1\,\sqrt{2}}{\sqrt{\pi}\,\alpha^{3/2}}\right)(\,R^4-(t-r)^4-\alpha^2-4\,\alpha\,(t-r)^2\,)$$
$$+\,2\,\pi\,\left(\frac{\%2\,\sqrt{2}}{\sqrt{\pi}\,\sqrt{\alpha}}-\frac{\%1\,\sqrt{2}}{\sqrt{\pi}\,\sqrt{\alpha}}\right)(\,-4\,(t-r)^3-8\,\alpha\,(t-r)\,)$$
$$+\,\pi\,\left(\Phi\left(\frac{1}{2}\,\frac{(\,(t-r)+R\,)\,\sqrt{2}}{\sqrt{\alpha}}\right)-\Phi\left(\frac{1}{2}\,\frac{(\,(t-r)-R\,)\,\sqrt{2}}{\sqrt{\alpha}}\right)\right)(\,-12\,(t-r)^2-8\,\alpha\,)$$
$$+\,2\,\alpha\,\%3\,\sqrt{2}\,\sqrt{\frac{\pi}{\alpha}}\,(\,(t-r)^2-R^2-3\,a\,)+6\,\alpha\,(\,\%2-\%1\,)\,\sqrt{2}\,\sqrt{\frac{\pi}{\alpha}}\,(t-r)+\alpha\,(t-r)$$
$$\left(-\frac{\%2}{\alpha}+\frac{(\,(t-r)+R\,)^2\,\%2}{\alpha^2}+\frac{\%1}{\alpha}-\frac{(\,(t-r)-R\,)^2\,\%1}{\alpha^2}\right)\sqrt{2}\,\sqrt{\frac{\pi}{\alpha}}\,(\,(t-r)^2-R^2-3\,\alpha\,)$$
$$+\,4\,\alpha\,(t-r)^2\,\%3\,\sqrt{2}\,\sqrt{\frac{\pi}{\alpha}}+\alpha\,R$$
$$\left(-\frac{\%2}{\alpha}+\frac{(\,(t-r)+R\,)^2\,\%2}{\alpha^2}-\frac{\%1}{\alpha}+\frac{(\,(t-r)-R\,)^2\,\%1}{\alpha^2}\right)\sqrt{2}\,\sqrt{\frac{\pi}{\alpha}}$$

$$\left((t-r)^2+\frac{1}{3}R^2+\alpha\right)+4\,\alpha\,R\left(-\frac{(\,(t-r)+R\,)\,\%2}{\alpha}-\frac{(\,(t-r)-R\,)\,\%1}{\alpha}\right)\sqrt{2}\sqrt{\frac{\pi}{\alpha}}\,(t-r)$$
$$+2\,\alpha\,R\,(\,\%2+\%1\,)\sqrt{2}\sqrt{\frac{\pi}{\alpha}}\Bigg)\Big/\left(\pi^2\,\alpha^2\,r\right) \tag{A.49}$$

$$\ddot{h}_d^{22}(\vec{x},t)=\frac{1}{60}G\sqrt{\frac{2\alpha}{\pi}}\left(\left(-\frac{(\,(t-r)+R\,)\,\%2\,\sqrt{2}}{\sqrt{\pi}\,\alpha^{3/2}}+\frac{(\,(t-r)-R\,)\,\%1\,\sqrt{2}}{\sqrt{\pi}\,\alpha^{3/2}}\right)\right.$$
$$(\,-R^4-5\,(t-r)^4-5\,\alpha^2-20\,\alpha\,(t-r)^2+6\,(t-r)^2\,R^2-4\,\alpha\,R^2\,)$$
$$+2\left(\frac{\%2\,\sqrt{2}}{\sqrt{\pi}\,\sqrt{\alpha}}-\frac{\%1\,\sqrt{2}}{\sqrt{\pi}\,\sqrt{\alpha}}\right)(\,-20\,(t-r)^3-40\,\alpha\,(t-r)+12\,(t-r)\,R^2\,)+$$
$$\left(\Phi\left(\frac{1}{2}\,\frac{(\,(t-r)+R\,)\,\sqrt{2}}{\sqrt{\alpha}}\right)-\Phi\left(\frac{1}{2}\,\frac{(\,(t-r)-R\,)\,\sqrt{2}}{\sqrt{\alpha}}\right)\right)(\,-60\,(t-r)^2-40\,\alpha+12\,R^2\,)$$
$$+18\,\sqrt{2}\sqrt{\frac{\alpha}{\pi}}\,(t-r)\,(\,\%2-\%1\,)+2\,\sqrt{2}\sqrt{\frac{\alpha}{\pi}}\,(\,3\,(t-r)^2+R^2-3\,\alpha\,)\,\%3$$
$$+12\,(t-r)^2\,\sqrt{2}\sqrt{\frac{\alpha}{\pi}}\,\%3+(t-r)\,\sqrt{2}\sqrt{\frac{\alpha}{\pi}}\,(\,3\,(t-r)^2+R^2-3\,\alpha\,)$$
$$\left(-\frac{\%2}{\alpha}+\frac{(\,(t-r)+R\,)^2\,\%2}{\alpha^2}+\frac{\%1}{\alpha}-\frac{(\,(t-r)-R\,)^2\,\%1}{\alpha^2}\right)$$
$$+18\,R\,(\,\%2+\%1\,)\,\sqrt{2}\sqrt{\frac{\alpha}{\pi}}$$
$$+36\,R\,(t-r)\left(-\frac{(\,(t-r)+R\,)\,\%2}{\alpha}-\frac{(\,(t-r)-R\,)\,\%1}{\alpha}\right)\sqrt{2}\sqrt{\frac{\alpha}{\pi}}+R\,\sqrt{2}\sqrt{\frac{\alpha}{\pi}}$$
$$\left.\left(9\,(t-r)^2+\frac{11}{3}\,R^2+5\,\alpha\right)\left(-\frac{\%2}{\alpha}+\frac{(\,(t-r)+R\,)^2\,\%2}{\alpha^2}-\frac{\%1}{\alpha}+\frac{(\,(t-r)-R\,)^2\,\%1}{\alpha^2}\right)\right)$$
$$\left(\sin(\,\alpha_0\,)^2\sin(\,\gamma_0\,)^2-1\right)\Big/(\,r\,\pi\,\alpha^2\,)+\frac{1}{36}G\sqrt{\frac{2\alpha}{\pi}}\Bigg($$
$$\pi\left(-\frac{(\,(t-r)+R\,)\,\%2\,\sqrt{2}}{\sqrt{\pi}\,\alpha^{3/2}}+\frac{(\,(t-r)-R\,)\,\%1\,\sqrt{2}}{\sqrt{\pi}\,\alpha^{3/2}}\right)(\,R^4-(t-r)^4-\alpha^2-4\,\alpha\,(t-r)^2\,)$$
$$+2\,\pi\left(\frac{\%2\,\sqrt{2}}{\sqrt{\pi}\,\sqrt{\alpha}}-\frac{\%1\,\sqrt{2}}{\sqrt{\pi}\,\sqrt{\alpha}}\right)(\,-4\,(t-r)^3-8\,\alpha\,(t-r)\,)$$
$$+\pi\left(\Phi\left(\frac{1}{2}\,\frac{(\,(t-r)+R\,)\,\sqrt{2}}{\sqrt{\alpha}}\right)-\Phi\left(\frac{1}{2}\,\frac{(\,(t-r)-R\,)\,\sqrt{2}}{\sqrt{\alpha}}\right)\right)(\,-12\,(t-r)^2-8\,\alpha\,)$$
$$+2\,\alpha\,\%3\,\sqrt{2}\sqrt{\frac{\pi}{\alpha}}\,(\,(t-r)^2-R^2-3\,a\,)+6\,\alpha\,(\,\%2-\%1\,)\,\sqrt{2}\sqrt{\frac{\pi}{\alpha}}\,(t-r)+\alpha\,(t-r)$$
$$\left(-\frac{\%2}{\alpha}+\frac{(\,(t-r)+R\,)^2\,\%2}{\alpha^2}+\frac{\%1}{\alpha}-\frac{(\,(t-r)-R\,)^2\,\%1}{\alpha^2}\right)\sqrt{2}\sqrt{\frac{\pi}{\alpha}}\,(\,(t-r)^2-R^2-3\,\alpha\,)$$
$$+4\,\alpha\,(t-r)^2\,\%3\,\sqrt{2}\sqrt{\frac{\pi}{\alpha}}+\alpha\,R$$
$$\left(-\frac{\%2}{\alpha}+\frac{(\,(t-r)+R\,)^2\,\%2}{\alpha^2}-\frac{\%1}{\alpha}+\frac{(\,(t-r)-R\,)^2\,\%1}{\alpha^2}\right)\sqrt{2}\sqrt{\frac{\pi}{\alpha}}$$
$$\left((t-r)^2+\frac{1}{3}R^2+\alpha\right)+4\,\alpha\,R\left(-\frac{(\,(t-r)+R\,)\,\%2}{\alpha}-\frac{(\,(t-r)-R\,)\,\%1}{\alpha}\right)\sqrt{2}\sqrt{\frac{\pi}{\alpha}}\,(t-r)$$

$$+ 2\,\alpha\,R\,(\,\%2 + \%1\,)\,\sqrt{2}\,\sqrt{\frac{\pi}{\alpha}}\Big) \Big/ \Big(\pi^2\,\alpha^2\,r\Big) \tag{A.50}$$

$$\ddot{h}_d^{33}(\vec{x},t) = \frac{1}{90} G \sqrt{\frac{2\alpha}{\pi}} \Bigg(\Bigg(-\frac{(\,(t-r)+R\,)\,\%2\,\sqrt{2}}{\sqrt{\pi}\,\alpha^{3/2}} + \frac{(\,(t-r)-R\,)\,\%1\,\sqrt{2}}{\sqrt{\pi}\,\alpha^{3/2}} \Bigg)$$
$$(\,-R^4 - 5\,(t-r)^4 - 5\,\alpha^2 - 20\,\alpha\,(t-r)^2 + 6\,(t-r)^2\,R^2 - 4\,\alpha\,R^2\,)$$
$$+ 2\,\left(\frac{\%2\,\sqrt{2}}{\sqrt{\pi}\,\sqrt{\alpha}} - \frac{\%1\,\sqrt{2}}{\sqrt{\pi}\,\sqrt{\alpha}}\right)\,(\,-20\,(t-r)^3 - 40\,\alpha\,(t-r) + 12\,(t-r)\,R^2\,) +$$
$$\left(\Phi\left(\frac{1}{2}\,\frac{(\,(t-r)+R\,)\,\sqrt{2}}{\sqrt{\alpha}}\right) - \Phi\left(\frac{1}{2}\,\frac{(\,(t-r)-R\,)\,\sqrt{2}}{\sqrt{\alpha}}\right)\right)\,(\,-60\,(t-r)^2 - 40\,\alpha + 12\,R^2\,)$$
$$+ 18\,\sqrt{2}\,\sqrt{\frac{\alpha}{\pi}}\,(t-r)\,(\,\%2 - \%1\,) + 2\,\sqrt{2}\,\sqrt{\frac{\alpha}{\pi}}\,(\,3\,(t-r)^2 + R^2 - 3\,\alpha\,)\,\%3$$
$$+ 12\,(t-r)^2\,\sqrt{2}\,\sqrt{\frac{\alpha}{\pi}}\,\%3 + (t-r)\,\sqrt{2}\,\sqrt{\frac{\alpha}{\pi}}\,(\,3\,(t-r)^2 + R^2 - 3\,\alpha\,)$$
$$\left(-\frac{\%2}{\alpha} + \frac{(\,(t-r)+R\,)^2\,\%2}{\alpha^2} + \frac{\%1}{\alpha} - \frac{(\,(t-r)-R\,)^2\,\%1}{\alpha^2}\right)$$
$$+ 18\,R\,(\,\%2 + \%1\,)\,\sqrt{2}\,\sqrt{\frac{\alpha}{\pi}}$$
$$+ 36\,R\,(t-r)\,\left(-\frac{(\,(t-r)+R\,)\,\%2}{\alpha} - \frac{(\,(t-r)-R\,)\,\%1}{\alpha}\right)\,\sqrt{2}\,\sqrt{\frac{\alpha}{\pi}} + R\,\sqrt{2}\,\sqrt{\frac{\alpha}{\pi}}$$
$$\left(9\,(t-r)^2 + \frac{11}{3}\,R^2 + 5\,\alpha\right)\,\left(-\frac{\%2}{\alpha} + \frac{(\,(t-r)+R\,)^2\,\%2}{\alpha^2} - \frac{\%1}{\alpha} + \frac{(\,(t-r)-R\,)^2\,\%1}{\alpha^2}\right)\Bigg)$$
$$\left(1 - \frac{3}{2}\,\sin(\,\gamma_0\,)^2\right) \Big/ (\,r\,\pi\,\alpha^2\,) + \frac{1}{36} G \sqrt{\frac{2\alpha}{\pi}} \Bigg($$
$$\pi\,\left(-\frac{(\,(t-r)+R\,)\,\%2\,\sqrt{2}}{\sqrt{\pi}\,\alpha^{3/2}} + \frac{(\,(t-r)-R\,)\,\%1\,\sqrt{2}}{\sqrt{\pi}\,\alpha^{3/2}}\right)\,(\,R^4 - (t-r)^4 - \alpha^2 - 4\,\alpha\,(t-r)^2\,)$$
$$+ 2\,\pi\,\left(\frac{\%2\,\sqrt{2}}{\sqrt{\pi}\,\sqrt{\alpha}} - \frac{\%1\,\sqrt{2}}{\sqrt{\pi}\,\sqrt{\alpha}}\right)\,(\,-4\,(t-r)^3 - 8\,\alpha\,(t-r)\,)$$
$$+ \pi\,\left(\Phi\left(\frac{1}{2}\,\frac{(\,(t-r)+R\,)\,\sqrt{2}}{\sqrt{\alpha}}\right) - \Phi\left(\frac{1}{2}\,\frac{(\,(t-r)-R\,)\,\sqrt{2}}{\sqrt{\alpha}}\right)\right)\,(\,-12\,(t-r)^2 - 8\,\alpha\,)$$
$$+ 2\,\alpha\,\%3\,\sqrt{2}\,\sqrt{\frac{\pi}{\alpha}}\,(\,(t-r)^2 - R^2 - 3\,a\,) + 6\,\alpha\,(\,\%2 - \%1\,)\,\sqrt{2}\,\sqrt{\frac{\pi}{\alpha}}\,(t-r) + \alpha\,(t-r)$$
$$\left(-\frac{\%2}{\alpha} + \frac{(\,(t-r)+R\,)^2\,\%2}{\alpha^2} + \frac{\%1}{\alpha} - \frac{(\,(t-r)-R\,)^2\,\%1}{\alpha^2}\right)\,\sqrt{2}\,\sqrt{\frac{\pi}{\alpha}}\,(\,(t-r)^2 - R^2 - 3\,\alpha\,)$$
$$+ 4\,\alpha\,(t-r)^2\,\%3\,\sqrt{2}\,\sqrt{\frac{\pi}{\alpha}} + \alpha\,R$$
$$\left(-\frac{\%2}{\alpha} + \frac{(\,(t-r)+R\,)^2\,\%2}{\alpha^2} - \frac{\%1}{\alpha} + \frac{(\,(t-r)-R\,)^2\,\%1}{\alpha^2}\right)\,\sqrt{2}\,\sqrt{\frac{\pi}{\alpha}}$$
$$\left((t-r)^2 + \frac{1}{3}\,R^2 + \alpha\right)\, + 4\,\alpha\,R\,\left(-\frac{(\,(t-r)+R\,)\,\%2}{\alpha} - \frac{(\,(t-r)-R\,)\,\%1}{\alpha}\right)\,\sqrt{2}\,\sqrt{\frac{\pi}{\alpha}}\,(t-r)$$
$$+ 2\,\alpha\,R\,(\,\%2 + \%1\,)\,\sqrt{2}\,\sqrt{\frac{\pi}{\alpha}}\Big) \Big/ \Big(\pi^2\,\alpha^2\,r\Big)$$

$$
\begin{aligned}
\%1 &:= \mathrm{e}^{\left(-1/2\,\frac{((t-r)-R)^2}{\alpha}\right)}\\
\%2 &:= \mathrm{e}^{\left(-1/2\,\frac{((t-r)+R)^2}{\alpha}\right)}\\
\%3 &:= -\frac{((t-r)+R)\,\%2}{\alpha}+\frac{((t-r)-R)\,\%1}{\alpha}
\end{aligned}
\tag{A.51}
$$

A.2 The gravitational force exerted on non-relativistic particles by a quantum string

In this section we will compute the gravitational force exerted on non-relativistic particles as given by eq.(6.28):

$$
\begin{aligned}
\mathcal{F}(\vec{x},t)\hat{x}^i = \frac{d^2x^i}{dt^2} &= 2G\left[\frac{8\pi}{3}\hat{x}^i\left(\frac{d\mathcal{G}_1}{dt}+r\frac{d^2\mathcal{G}_1}{dt^2}\right)\right.\\
&\left.-\frac{dh_a^{00}}{dr}\,\hat{x}^i-\frac{1}{2}\frac{d\ddot{h}_d^{00}}{dr}\,\hat{x}^i\right].
\end{aligned}
\tag{A.52}
$$

$h^{\mu\nu}(\vec{x},t)$ is given by eq.(5.11) and eqs.(5.16)-(5.36).

$$
\begin{aligned}
\mathcal{F}(\vec{x},t)\hat{x}^i = \frac{d^2x^i}{dt^2} &= \frac{8}{3}G\,\pi\,x^i\left(\frac{1}{8}\sqrt{2}\sqrt{\frac{\alpha}{\pi}}\left(\left(\frac{\%2\,\sqrt{2}}{\sqrt{\pi}\,\sqrt{\alpha}}-\frac{\%1\,\sqrt{2}}{\sqrt{\pi}\,\sqrt{\alpha}}\right)(t_{ret}\,R^2-{t_{ret}}^3-2\,\alpha\,t_{ret})\right.\right.\\
&+\%3\,(R^2-3\,{t_{ret}}^2-2\,\alpha)-\alpha\left(-\frac{(t_{ret}+R)\,\%2}{\alpha}+\frac{(t_{ret}-R)\,\%1}{\alpha}\right)\sqrt{2}\sqrt{\frac{\alpha}{\pi}}\\
&+R\,(\%2+\%1)\,\sqrt{2}\sqrt{\frac{\alpha}{\pi}}\\
&\left.+R\,t_{ret}\left(-\frac{(t_{ret}+R)\,\%2}{\alpha}-\frac{(t_{ret}-R)\,\%1}{\alpha}\right)\sqrt{2}\sqrt{\frac{\alpha}{\pi}}\right)\Big/\left(\pi^2\,\alpha^2\,r^2\right)+\frac{1}{8}\sqrt{2}\\
&\sqrt{\frac{\alpha}{\pi}}\left(\left(-\frac{(t_{ret}+R)\,\%2\,\sqrt{2}}{\sqrt{\pi}\,\alpha^{3/2}}+\frac{(t_{ret}-R)\,\%1\,\sqrt{2}}{\sqrt{\pi}\,\alpha^{3/2}}\right)(t_{ret}\,R^2-{t_{ret}}^3-2\,\alpha\,t_{ret})\right.\\
&+2\left(\frac{\%2\,\sqrt{2}}{\sqrt{\pi}\,\sqrt{\alpha}}-\frac{\%1\,\sqrt{2}}{\sqrt{\pi}\,\sqrt{\alpha}}\right)(R^2-3\,{t_{ret}}^2-2\,\alpha)-6\,\%3\,t\\
&-\alpha\left(-\frac{\%2}{\alpha}+\frac{(t_{ret}+R)^2\,\%2}{\alpha^2}+\frac{\%1}{\alpha}-\frac{(t_{ret}-R)^2\,\%1}{\alpha^2}\right)\sqrt{2}\sqrt{\frac{\alpha}{\pi}}\\
&+2\,R\left(-\frac{(t_{ret}+R)\,\%2}{\alpha}-\frac{(t_{ret}-R)\,\%1}{\alpha}\right)\sqrt{2}\sqrt{\frac{\alpha}{\pi}}\\
&\left.+R\,t_{ret}\left(-\frac{\%2}{\alpha}+\frac{(t_{ret}+R)^2\,\%2}{\alpha^2}-\frac{\%1}{\alpha}+\frac{(t_{ret}-R)^2\,\%1}{\alpha^2}\right)\sqrt{2}\sqrt{\frac{\alpha}{\pi}}\right)\Big/\left(r\,\pi^2\right.\\
&\left.\left.\alpha^2\right)\right)+\frac{1}{24}G\,x^i\,\sqrt{2}\sqrt{\frac{\alpha}{\pi}}\Bigg(\\
&\pi\left(-\frac{(t_{ret}+R)\,\%2\,\sqrt{2}}{\sqrt{\pi}\,\alpha^{3/2}}+\frac{(t_{ret}-R)\,\%1\,\sqrt{2}}{\sqrt{\pi}\,\alpha^{3/2}}\right)(R^4-{t_{ret}}^4-\alpha^2-4\,\alpha\,{t_{ret}}^2)
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\,\pi\,\left(\frac{\%2\,\sqrt{2}}{\sqrt{\pi}\,\sqrt{\alpha}} - \frac{\%1\,\sqrt{2}}{\sqrt{\pi}\,\sqrt{\alpha}}\right)\,(\,-4\,{t_{ret}}^3 - 8\,\alpha\,t_{ret}\,) + \pi\,\%3\,(\,-12\,{t_{ret}}^2 - 8\,\alpha\,)\\
&+ 2\,\alpha\,\left(-\frac{(\,t_{ret} + R\,)\,\%2}{\alpha} + \frac{(\,t_{ret} - R\,)\,\%1}{\alpha}\right)\,\sqrt{2}\,\sqrt{\frac{\pi}{\alpha}}\,(\,{t_{ret}}^2 - R^2 - 3\,\alpha\,)\\
&+ 6\,\alpha\,(\,\%2 - \%1\,)\,\sqrt{2}\,\sqrt{\frac{\pi}{\alpha}}\,t_{ret} + \alpha\,t_{ret}\\
&\left(-\frac{\%2}{\alpha} + \frac{(\,t_{ret} + R\,)^2\,\%2}{\alpha^2} + \frac{\%1}{\alpha} - \frac{(\,t_{ret} - R\,)^2\,\%1}{\alpha^2}\right)\,\sqrt{2}\,\sqrt{\frac{\pi}{\alpha}}\,(\,{t_{ret}}^2 - R^2 - 3\,\alpha\,)\\
&+ 4\,\alpha\,{t_{ret}}^2\,\left(-\frac{(\,t_{ret} + R\,)\,\%2}{\alpha} + \frac{(\,t_{ret} - R\,)\,\%1}{\alpha}\right)\,\sqrt{2}\,\sqrt{\frac{\pi}{\alpha}} + \alpha\,R\\
&\left(-\frac{\%2}{\alpha} + \frac{(\,t_{ret} + R\,)^2\,\%2}{\alpha^2} - \frac{\%1}{\alpha} + \frac{(\,t_{ret} - R\,)^2\,\%1}{\alpha^2}\right)\,\sqrt{2}\,\sqrt{\frac{\pi}{\alpha}}\\
&\left({t_{ret}}^2 + \frac{1}{3}\,R^2 + \alpha\right)\, + 4\,\alpha\,R\,\left(-\frac{(\,t_{ret} + R\,)\,\%2}{\alpha} - \frac{(\,t_{ret} - R\,)\,\%1}{\alpha}\right)\,\sqrt{2}\,\sqrt{\frac{\pi}{\alpha}}\,t_{ret}\\
&+ 2\,\alpha\,R\,(\,\%2 + \%1\,)\,\sqrt{2}\,\sqrt{\frac{\pi}{\alpha}}\Bigg)\Big/\left(\pi^2\,\alpha^2\,r^2\right)\\
&\%1 = \mathrm{e}^{\left(-1/2\,\frac{(\,t_{ret} - R\,)^2}{\alpha}\right)}\\
&\%2 = \mathrm{e}^{\left(-1/2\,\frac{(\,t_{ret} + R\,)^2}{\alpha}\right)}\\
&\%3 = \Phi\left(\frac{1}{2}\,\frac{(\,t_{ret} + R\,)\,\sqrt{2}}{\sqrt{\alpha}}\right) - \Phi\left(\frac{1}{2}\,\frac{(\,t_{ret} - R\,)\,\sqrt{2}}{\sqrt{\alpha}}\right)
\end{aligned}
\tag{A.53}
$$

APPENDIX B

TABLE OF INTEGRALS AND FUNCTIONS MOST USED IN THIS WORK

This appendix is intended to provide the reader with some of the useful technical content used in this work in order to avoid otherwise necessary recourse to the appropriate references [88, 89] or to consulting special packages such as Maple.

B.1 Integrals

$$\int_0^{\pi} \sin\gamma \, e^{-iEr\cos\gamma} \, d\gamma = \frac{2}{Er}\sin Er \tag{B.1}$$

$$\int_0^{\pi} \sin\gamma \, e^{iEr\cos\gamma} \, d\gamma = \frac{2}{Er}\sin Er \tag{B.2}$$

$$\int_0^{\pi} \sin^3\gamma \, e^{-iEr\cos\gamma} \, d\gamma = -\frac{4}{E^2r^2}\left(\cos Er - \frac{\sin Er}{Er}\right) \tag{B.3}$$

$$\int_0^{\pi} \cos\gamma\sin\gamma \, e^{-iEr\cos\gamma} \, d\gamma = \frac{2i}{E^2r^2}\left(Er\cos Er - \sin Er\right) \tag{B.4}$$

$$\int_0^{\pi} \cos^2\gamma\sin\gamma \, e^{-iEr\cos\gamma} \, d\gamma = 2\frac{\sin Er}{Er} + \frac{4}{E^2r^2}\left(\cos Er - \frac{\sin Er}{Er}\right) \tag{B.5}$$

$$\int_0^{2\pi} e^{-iE\rho\cos\alpha\sin\gamma} \, d\alpha = 2\pi J_0(E\rho\sin\gamma) \tag{B.6}$$

$$\int_0^{2\pi} e^{iE\rho\cos\alpha\sin\gamma} \, d\alpha = 2\pi J_0(E\rho\sin\gamma) \tag{B.7}$$

$$\int_0^{2\pi} \cos\alpha \; e^{-iE\rho\cos\alpha\sin\gamma} \; d\alpha \quad = \quad -2\pi i J_1(E\rho\sin\gamma) \tag{B.8}$$

$$\int_0^{2\pi} \cos\alpha \; e^{iE\rho\cos\alpha\sin\gamma} \; d\alpha \quad = \quad 2\pi i J_1(E\rho\sin\gamma) \tag{B.9}$$

$$\int_0^{2\pi} \cos^2\alpha \; e^{-iE\rho\cos\alpha\sin\gamma} \; d\alpha \quad = \quad \pi\left(J_0(E\rho\sin\gamma) - J_2(E\rho\sin\gamma)\right) \tag{B.10}$$

$$\int_0^{2\pi} \cos^2\alpha \; e^{iE\rho\cos\alpha\sin\gamma} \; d\alpha \quad = \quad \pi\left(J_0(E\rho\sin\gamma) - J_2(E\rho\sin\gamma)\right) \tag{B.11}$$

$$\int_0^{2\pi} \sin\alpha \; e^{-iE\rho\cos\alpha\sin\gamma} \; d\alpha = \int_0^{2\pi} \sin\alpha \; e^{iE\rho\cos\alpha\sin\gamma} \; d\alpha \quad = \quad 0 \tag{B.12}$$

$$\int_0^{2\pi} \sin\alpha\cos\alpha \; e^{-iE\rho\cos\alpha\sin\gamma} \; d\alpha = \int_0^{2\pi} \sin\alpha\cos\alpha \; e^{iE\rho\cos\alpha\sin\gamma} \; d\alpha \quad = \quad 0 \tag{B.13}$$

$$\int_0^{2\pi} \sin^2\alpha \; e^{-iE\rho\cos\alpha\sin\gamma} \; d\alpha \quad = \quad \pi\left(J_0(E\rho\sin\gamma) + J_2(E\rho\sin\gamma)\right) \tag{B.14}$$

$$\int_0^{2\pi} \sin^2\alpha \; e^{iE\rho\cos\alpha\sin\gamma} \; d\alpha \quad = \quad \pi\left(J_0(E\rho\sin\gamma) + J_2(E\rho\sin\gamma)\right) \tag{B.15}$$

$$\int e^{-(ax^2+2bx)} \; dx \quad = \quad \frac{e^{\frac{b^2}{a}}}{2}\sqrt{\frac{\pi}{a}}\Phi(\sqrt{a}x + \frac{b}{\sqrt{a}}) \tag{B.16}$$

$$\int \Phi(x) \; dx = x\Phi(x) + \frac{e^{-x^2}}{\sqrt{\pi}} \tag{B.17}$$

$$\int \Phi(\sqrt{a}x + \frac{b}{\sqrt{a}}) \; dx = (x + \frac{b}{a})\Phi(\sqrt{a}x + \frac{b}{\sqrt{a}}) + \frac{e^{-(ax^2+2bx+\frac{b^2}{a})}}{\sqrt{\pi a}} \tag{B.18}$$

$$\int xe^{-(ax^2+2bx)}\,dx = -\frac{e^{-(ax^2+2bx)}}{2a} - \frac{e^{\frac{b^2}{a}}}{2}\sqrt{\frac{\pi}{a}}\frac{b}{a}\Phi(\sqrt{a}x + \frac{b}{\sqrt{a}}) \tag{B.19}$$

$$\begin{aligned}\int x^2e^{-(ax^2+2bx)}\,dx &= -\frac{x}{2a}e^{-(ax^2+2bx)} + \frac{e^{\frac{b^2}{a}}}{4}\sqrt{\frac{\pi}{a}}\frac{1}{a}\Phi(\sqrt{a}x + \frac{b}{\sqrt{a}}) + \\ &\quad \sqrt{\frac{\pi}{a}}\frac{b^2}{2a^2}e^{\frac{b^2}{a}}\Phi(\sqrt{a}x + \frac{b}{\sqrt{a}}) + \frac{b}{2a^2}e^{-(ax^2+2bx)}\end{aligned} \tag{B.20}$$

$$\begin{aligned}\int x^3e^{-(ax^2+2bx)}\,dx &= -\frac{x^2}{2a}e^{-(ax^2+2bx)} - b\frac{e^{\frac{b^2}{a}}}{2a^2}\sqrt{\frac{\pi}{a}}\Phi(\sqrt{a}x + \frac{b}{\sqrt{a}}) - \\ &\quad \sqrt{\frac{\pi}{a}}\frac{b^3}{2a^3}e^{\frac{b^2}{a}}\Phi(\sqrt{a}x + \frac{b}{\sqrt{a}}) - \sqrt{\frac{\pi}{a}}\frac{b}{4a^2}e^{\frac{b^2}{a}}\Phi(\sqrt{a}x + \frac{b}{\sqrt{a}}) - \\ &\quad \frac{b^2}{2a^3}e^{-(ax^2+2bx)} + \frac{1}{2a^2}e^{-(ax^2+2bx)}(bx-1)\end{aligned} \tag{B.21}$$

$$\begin{aligned}\int x^4e^{-(ax^2+2bx)}\,dx &= -\frac{x^3}{2a}e^{-(ax^2+2bx)} + \frac{bx^2}{2a^2}e^{-(ax^2+2bx)} - \frac{3x}{4a^2}e^{-(ax^2+2bx)} - \\ &\quad \frac{b^2x}{2a^3}e^{-(ax^2+2bx)} + \frac{5b}{4a^3}e^{-(ax^2+2bx)} + \frac{b^3}{2a^4}e^{-(ax^2+2bx)} + \\ &\quad \sqrt{\frac{\pi}{a}}\frac{3}{8a^2}e^{\frac{b^2}{a}}\Phi(\sqrt{a}x + \frac{b}{\sqrt{a}}) + \sqrt{\frac{\pi}{a}}\frac{b^4}{2a^4}e^{\frac{b^2}{a}}\Phi(\sqrt{a}x + \frac{b}{\sqrt{a}}) + \\ &\quad \sqrt{\frac{\pi}{a}}\frac{6b^2}{4a^3}e^{\frac{b^2}{a}}\Phi(\sqrt{a}x + \frac{b}{\sqrt{a}})\end{aligned} \tag{B.22}$$

$$\int x^5e^{-\frac{x^2}{2\alpha}}\,dx = -e^{-\frac{x^2}{2\alpha}}\alpha\left(x^4 + 4\alpha x^2 + 8\alpha^2\right) \tag{B.23}$$

$$\int x^6e^{-\frac{x^2}{2\alpha}}\,dx = \frac{15\alpha^3}{2}\sqrt{2\alpha\pi}\Phi(\frac{x}{\sqrt{2\alpha}}) - e^{-\frac{x^2}{2\alpha}}x\alpha\left(x^4 + 5\alpha x^2 + 15\alpha^2 x\right) \tag{B.24}$$

$$\int x\Phi(\frac{x}{\sqrt{2\alpha}})\,dx = \frac{x^2}{2}\Phi(\frac{x}{\sqrt{2\alpha}}) - \frac{\alpha}{2}\Phi(\frac{x}{\sqrt{2\alpha}}) - \frac{\alpha x}{2}\sqrt{\frac{2}{\alpha\pi}}e^{-\frac{x^2}{2\alpha}} \tag{B.25}$$

$$\int x^2\Phi(\frac{x}{\sqrt{2\alpha}})\,dx = \frac{x^3}{3}\Phi(\frac{x}{\sqrt{2\alpha}}) + \sqrt{\frac{2\alpha}{\pi}}e^{-\frac{x^2}{2\alpha}}\left(\frac{x^2}{3} + \frac{2\alpha}{3}\right) \tag{B.26}$$

$$\int x^3 \Phi(\frac{x}{\sqrt{2\alpha}})\, dx = \frac{1}{4}\Phi(\frac{x}{\sqrt{2\alpha}})\left[x^4 - 3\alpha^2\right] + \frac{u}{4}\sqrt{\frac{2\alpha}{\pi}} e^{-\frac{x^2}{2\alpha}}\left(u^2 + 3\alpha\right) \tag{B.27}$$

$$\int_0^\infty e^{-\frac{\alpha x^2}{2}}\, dx = \frac{1}{2}\sqrt{\frac{2\pi}{\alpha}} \tag{B.28}$$

$$\int_0^\infty x^2 e^{-\frac{\alpha x^2}{2}}\, dx = \frac{3}{2\alpha}\sqrt{\frac{2\pi}{\alpha}} \tag{B.29}$$

$$\int_0^\infty x^4 e^{-\frac{\alpha x^2}{2}}\, dx = \frac{3}{2\alpha^2}\sqrt{\frac{2\pi}{\alpha}} \tag{B.30}$$

$$\int_{-\infty}^\infty e^{-\frac{\alpha x^2}{2}} x \sin xy\, dx = \sqrt{\frac{\pi}{2}}\frac{e^{-\frac{y^2}{4\alpha}}}{\alpha}\left[D_1\left(-\frac{y}{\sqrt{\alpha}}\right) - D_1\left(\frac{y}{\sqrt{\alpha}}\right)\right] \tag{B.31}$$

$$\int_{-\infty}^\infty e^{-\frac{\alpha x^2}{2}} x^3 \sin xy\, dx = \sqrt{\frac{\pi}{2}}\frac{e^{-\frac{y^2}{4\alpha}}}{\alpha^2}\left[D_3\left(-\frac{y}{\sqrt{\alpha}}\right) - D_3\left(\frac{y}{\sqrt{\alpha}}\right)\right] \tag{B.32}$$

$$\int_{-\infty}^\infty \frac{e^{-\frac{\alpha x^2}{2}}}{x} \sin xy\, dx = y\sqrt{\frac{2\pi}{\alpha}} e^{-\frac{y^2}{4\alpha}}\ {}_1F_1\left(1; 3/2; \frac{y^2}{2\alpha}\right) \tag{B.33}$$

$$\int_0^\infty e^{-\alpha x^2}\ \sin xy\, dx = \frac{y}{2\alpha}\ e^{-\frac{y^2}{4\alpha}}\ {}_1F_1(1/2; 3/2; \frac{y^2}{4\alpha}) \tag{B.34}$$

$$\int_0^\infty x\ e^{-\alpha x^2}\ \sin xy\, dx = \frac{y}{4\alpha}\sqrt{\frac{\pi}{\alpha}}\ e^{-\frac{y^2}{4\alpha}} \tag{B.35}$$

$$\int_0^\infty e^{-\alpha x^2}\ \frac{\sin xy}{x}\, dx = \frac{\pi}{2}\Phi(\frac{y}{2\sqrt{\alpha}}) \tag{B.36}$$

$$\int_0^\infty x^2\ e^{-\alpha x^2}\ \sin xy\, dx = \frac{y}{2\alpha^2}\ e^{-\frac{y^2}{4\alpha}}\ {}_1F_1(-1/2; 3/2; \frac{y^2}{4\alpha}) \tag{B.37}$$

$$\int_0^\infty e^{-\alpha x^2} \cos xy \, dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{y^2}{4\alpha}} \, {}_1F_1(-1/2; 3/2; \frac{y^2}{4\alpha}) \tag{B.38}$$

$$\int_0^\infty x \, e^{-\alpha x^2} \cos xy \, dx = \frac{1}{2\alpha} \, {}_1F_1(1; 1/2; -\frac{y^2}{4\alpha}) \tag{B.39}$$

$$\int_0^\infty e^{-\alpha x^2} \frac{\cos xy}{x} \, dx = \int du_1 F_1(1; 3/2; -u) \tag{B.40}$$

$$\int_0^\infty x^2 \, e^{-\alpha x^2} \cos xy \, dx = \frac{1}{8\alpha^2}\sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{y^2}{4\alpha}} \, (2\alpha - y^2) \tag{B.41}$$

$$\int_0^\infty \sin(bt) \, t^{z-1} \, dt = \frac{\Gamma(z)}{b^z} \sin(\frac{\pi}{2} z) \tag{B.42}$$

$$\int_0^\infty \frac{\sin(bt)}{t} \, dt = \frac{\pi}{2} \, sign(b) \tag{B.43}$$

B.2 Special Functions

B.2.1 The Probability function

The probability function is defined as

$$\Phi(z) \equiv \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt \tag{B.44}$$

and it satisfies

$$\Phi(z) \overset{z \to \pm\infty}{=} 1 - \frac{e^{-z^2}}{\sqrt{\pi} z} \tag{B.45}$$

$$\Phi(z) \overset{z \to 0}{=} \frac{2z}{\sqrt{\pi}} \tag{B.46}$$

$$\Phi(-z) = -\Phi(z). \tag{B.47}$$

Its derivative is given by

$$\frac{d\Phi(z)}{dz} = \frac{2}{\sqrt{\pi}} \, e^{-z^2} \tag{B.48}$$

B.2.2 The Whittaker functions

The Whittaker functions are defined as:

$$\begin{aligned} D_p(z) &= 2^{\frac{p}{2}} e^{-\frac{z^2}{4}} \left\{ \frac{\sqrt{\pi}}{\Gamma\left(\frac{1-p}{2}\right)} \, {}_1F_1\left(-\frac{p}{2}, \frac{1}{2}; \frac{z^2}{2}\right) \right. \\ & \left. -\frac{\sqrt{2\pi}\, z}{\Gamma\left(-\frac{p}{2}\right)} \, {}_1F_1\left(\frac{1-p}{2}, \frac{3}{2}; \frac{z^2}{2}\right) \right\}. \end{aligned} \tag{B.49}$$

or

$$D_n(z) = 2^{-\frac{n}{2}} e^{-\frac{z^2}{4}} H_n\left(\frac{z}{\sqrt{2}}\right) \tag{B.50}$$

if $n \geq 0$ where $H_n(z)$ are the Hermite polynomials of order n. The assymptotic expansions of the Whittaker functions when $z \gg 1$, $z \gg p$ are:

$$\begin{aligned} D_p(z) \sim\ & e^{-\frac{z^2}{4}} z^p \left(1 - \frac{p(p-1)}{2z^2} + \frac{p(p-1)(p-2)(p-3)}{2 \cdot 4z^4} - \ldots\right) \\ & -\frac{\sqrt{2\pi}}{\Gamma(-p)} e^{p\pi i} z^{-p-1} \left(1 + \frac{(p+1)(p+2)}{2z^2} + \frac{(p+1)(p+2)(p+3)(p+4)}{2 \cdot 4z^4} + \ldots\right) \end{aligned} \tag{B.51}$$

if $\pi/4 < \arg z < 5\pi/4$ and

$$\begin{aligned} D_p(z) \sim\ & e^{-\frac{z^2}{4}} z^p \left(1 - \frac{p(p-1)}{2z^2} + \frac{p(p-1)(p-2)(p-3)}{2 \cdot 4z^4} - \ldots\right) \\ & -\frac{\sqrt{2\pi}}{\Gamma(-p)} e^{-p\pi i} z^{-p-1} \left(1 + \frac{(p+1)(p+2)}{2z^2} + \frac{(p+1)(p+2)(p+3)(p+4)}{2 \cdot 4z^4} + \ldots\right) \end{aligned} \tag{B.52}$$

if $-\pi/4 > \arg z > -5\pi/4$.

B.2.3 The Confluent Hypergeometric function

The confluent hypergeometric function satisfies the *Kummer's differential equation*:

$$z\frac{d^2w}{dz^2} + (b-z)\frac{dw}{dz} - \alpha w = 0$$

and it can be written as

$${}_1F_1\,(a; b; z) = 1 + \frac{az}{b} + \frac{(a)_2 z^2}{(a)_2 2!} + \ldots + \frac{(a)_n z^n}{(a)_n n!} + \ldots \tag{B.53}$$

where

$$(a)_n = a(a+1)(a+2)\ldots(a+n-1),$$

$$(a)_0 = 1.$$

The asymptotic behaviour of the confluent hypergeometric function is given by the following relations:

$$ {}_1F_1\,(a; b; z) = \frac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b} \quad \mathcal{R}z > 0 \tag{B.54}$$

$$ {}_1F_1\,(a; b; z) = \frac{\Gamma(b)}{\Gamma(b-a)} (-z)^{-a} \quad \mathcal{R}z < 0. \tag{B.55}$$

And we also have the following relation:

$$ {}_1F_1\left(1; 3/2; z^2\right) \;=\; e^{z^2}\, {}_1F_1\left(1/2; 3/2; -z^2\right). \tag{B.56}$$

The confluent hypergeometric function is also related to the probability function in the following way:

$$ {}_1F_1\left(1/2; 3/2; -z^2\right) \;=\; \frac{\sqrt{\pi}}{2z} \Phi(z). \tag{B.57}$$